Pattern distribution in faro words and permutations

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Faro words, $\mathcal{S}_{n,k}$

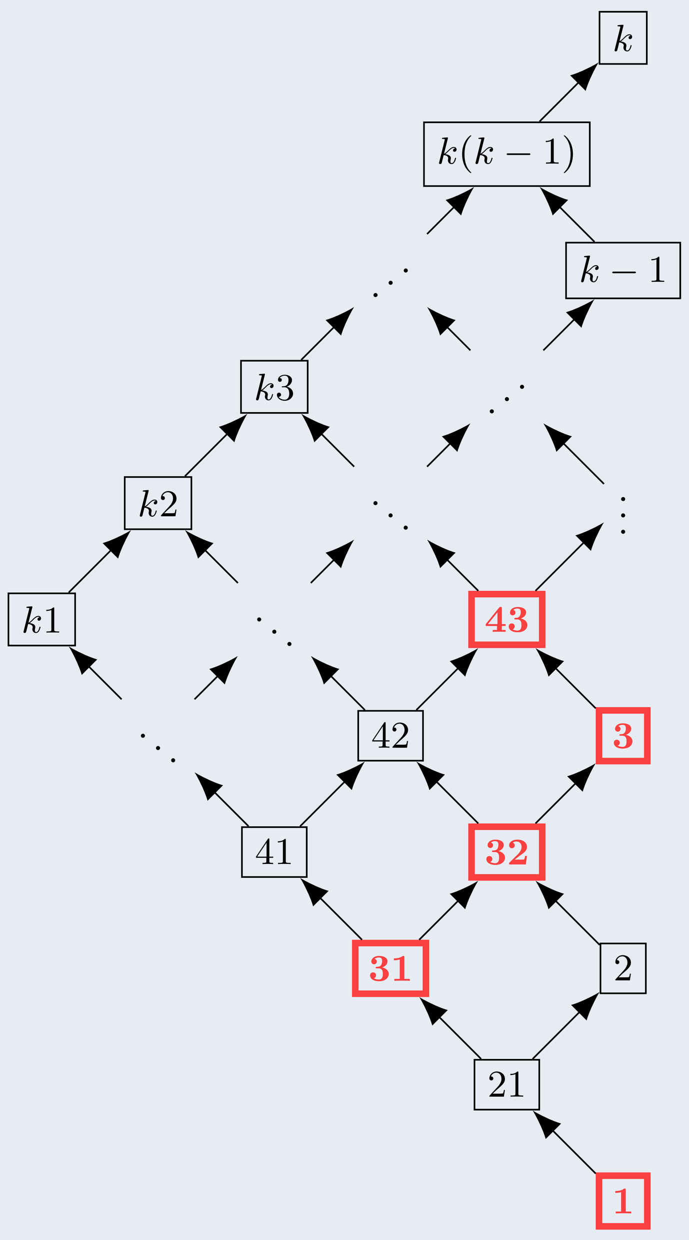
A *faro word* is an n-length k-ary word $w = w_1w_2\dots w_n$, $w_i \in [1,k]$ equal to a faro shuffle $w = u_1v_1u_2v_2u_3v_3\dots$ of two non-decreasing words $u = u_1u_2\dots$ and $v = v_1v_2\dots$ such that $0 \leqslant |u| - |v| \leqslant 1$ and $|u| + |v| = n$.

Faro words are characterised by the property $w_i \leqslant w_{i+2}$, $i \in [1, n-2]$.

Let $\mathcal{S}_{n,k}$ denote the set of k-ary faro words of length n.
Example: $\mathcal{S}_{4,2} = \{1111, 1112, 1121, 1122, 1212, 1222, 2121, 2122, 2222\}$.

$$|\mathcal{S}_{n,k}| = \binom{\lfloor \frac{n}{2} \rfloor + k - 1}{k-1}\binom{\lceil \frac{n}{2} \rceil + k - 1}{k-1}.$$

Decomposition into descent pairs and singletons



Blocks of a q-ary faro word w:

- **descent pair** is a consecutive subword w_iw_{i+1}, such that $w_i > w_{i+1}$.
- **singleton** in a faro word is a letter w_i standing outside any descent pair.

Order relation on all possible descent pairs and singletons: $q \preceq p$ iff qp is a k-ary faro word different from a descent pair.

Theorem. k-ary faro words are in bijection with all multichains of the lattice.

Example 1. The multichain $1 \preceq 1 \prec 31 \prec 32 \preceq 32 \prec 3 \prec 43$ corresponds to the faro word

$$11313232343 = (1)^2(31)(32)^2(3)(43).$$

Faro permutations, $\mathcal{P}_n$

Let $\mathcal{P}_n$ be the set of *faro permutations* of length n, *i.e.*, the set of permutations in $\mathcal{S}_{n,n}$. Obviously we have $\mathcal{P}_n \subset Av_n(321)$ and $|\mathcal{P}_n| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. For instance, $\mathcal{P}_3 = \{123, 132, 213\}$.

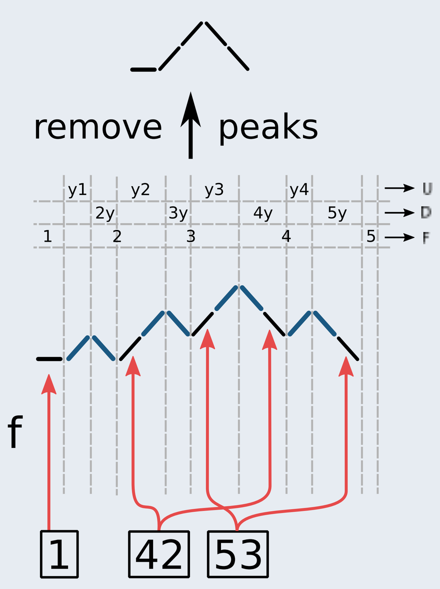
A **bijection** $g : \mathcal{P}_n \to \mathcal{B}_n$

1. Apply bijection f.
2. Remove peaks (i.e. patterns UD).

Transport of consecutive patterns

Faro permutation	Dispersed Dyck path
21	$\mathbf{U}$
12	$\mathbf{DU + DD + DF + FF + FU}$
132	$\mathbf{UU + DU + FU}$
213	$\mathbf{DU + DD + DF}$
123	$\sum_{\lambda,\mu \in \{F,U,D\}} \boldsymbol{\lambda} \mathbf{F} \boldsymbol{\mu}$

Exemple: the image by g of the permutation 14253 :



Examples: O.g.f. with respect to the length and the number of descents

$$\frac{2}{\sqrt{-4\,x^2y + 1} - 2\,x + 1}$$

Descent popularity on $\mathcal{P}_n$: $u_n = \frac{n+1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} - 2^{n-1}$ **(OEIS A107373)**

$$x^2 + 2\,x^3 + 7\,x^4 + 14\,x^5 + 38\,x^6 + 76\,x^7 + 187\,x^8 + 374\,x^9 + 874\,x^{10} + 1748\,x^{11} + 3958\,x^{12} + \dots$$

O.g.f. with respect to the length and the number of 213 **occurrences**

$$\frac{(y-1)\sqrt{-4\,x^2y+1} + y + 1}{y\left(\sqrt{-4\,x^2y+1} - 2\,x + 1\right)}$$

Popularity of 213: $x^3 + 4\,x^4 + 10\,x^5 + 28\,x^6 + 61\,x^7 + \dots$ **(New)**

- **faro involutions are counted by the Fibonacci numbers**
- **faro derangements are counted by the Catalan numbers**

Dispersed Dyck paths, $\mathcal{B}_n$

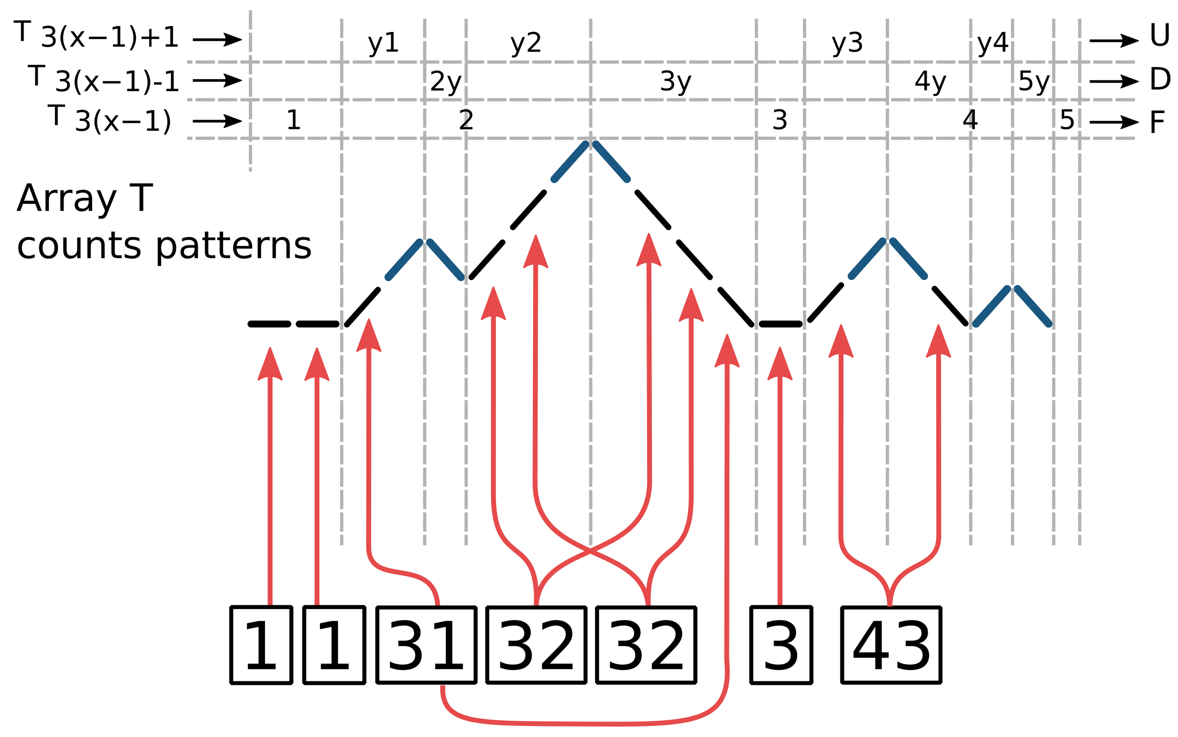
A *dispersed Dyck path* of length $n \geqslant 0$ is a lattice path that starts at $(0,0)$, ends at $(n,0)$, consisting of a sequence of flat $F = (1,0)$, up $U = (1,1)$ and down $D = (1,-1)$ steps, while always remaining in first quadrant and with all flat steps lying only on the x-axis. We denote by $\mathcal{B}_n$ the set of n-length dispersed Dyck path and we have $|\mathcal{B}_n| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Dispersed Dyck paths of length n are in trivial bijection with n-length prefixes of Dyck paths, also known as ballot paths.

Bijection f between $\mathcal{S}_{n,k}$ and $\mathcal{B}_{n+2(k-1),k-1}$

Theorem. There is a bijection f from $\mathcal{S}_{n,k}$ to the set $\mathcal{B}_{n+2(k-1),k-1}$ of dispersed Dyck paths of length $n + 2(k-1)$ having exactly $k-1$ peaks.

Example: the image by f of the 5-ary word 11313232343 :



1. For a given $w \in \mathcal{S}_{n,k}$, initialise $T_i := 0$ for $i \in [0, 3(k-1)]$.
2. Let $T_{3(x-1)}$ be the number of occurrences of singleton x in the decomposition of w
3. Let $T_{3(x-1)-1}$ be one plus the number of descent pairs xy in w
4. Let $T_{3(x-1)+1}$ be one plus the number of descent pairs yx in w
5. Construct a dispersed Dyck path $f(w)$ using the array T as follows:

$$f(w) = F^{T_0}U^{T_1}D^{T_2}F^{T_3}\dots F^{T_{3(k-2)}}U^{T_{3(k-2)+1}}D^{T_{3(k-2)+2}}F^{T_{3(k-1)}}.$$

Transport of consecutive patterns

- α^+ means one or more consecutive patterns α
- α^* means zero or more consecutive patterns α.

Faro word	Dispersed Dyck path
11	$\mathbf{FF}$
21	$\mathbf{UU}$
12	$\mathbf{DD(UD)^*UU + DD(UD)^*D +}$ $\mathbf{DD(UD)^*F + F(UD)^+F + F(UD)^*UU}$
111	$\mathbf{FFF}$
112	$\mathbf{FF(UD)^+F + FF(UD)^*UU}$
122	$\mathbf{F(UD)^+FF + DD(UD)^*FF}$
121	$\mathbf{FUU + UUU}$
212	$\mathbf{DDF + DDD}$
132	$\mathbf{F(UD)^+UU + U(UD)^+UU + DD(UD)^*UU}$
213	$\mathbf{DD(UD)^+F + DD(UD)^+D + DD(UD)^*UU}$
123	$\mathbf{DD(UD)^*F(UD)^*UU + DD(UD)^*F(UD)^+F}$ $\mathbf{+F(UD)^+F(UD)^*UU + F(UD)^+F(UD)^+F}$

Example: O.g.f. for faro words in $\mathcal{S}_{n,k}$ avoiding 11

$$\frac{2y\,(x+1)}{1 - y - x + x^2 - xy + x^3 + (x+1)\sqrt{x^4 - 2\,x^2y - 2\,x^2 + y^2 - 2\,y + 1}}$$

For $k = 3$, the sequence seems to be OEIS A004116: $u_n = \left\lfloor \frac{n^2+6n-3}{4} \right\rfloor$

Open questions

- We know that the diagonal set $\mathcal{S}_{n,n}$ is enumerated by $u_0 = 1, u_1 = 2$ and

  $$9n(3n+2)(3n-1)(18n^3+51n^2+41n+6)(1+3n)^2u_n + 36n(n+1)(54n^5+90n^4-87n^3-217n^2-112n-20)u_{n+1} = 16n(n+3)(n+2)(18n^3-3n^2-7n-2)(n+1)^2u_{n+2}$$

  Open question: Can we obtain similar results for all other diagonal sets?
- **Conjecture:** $|\text{Subexcedent in } \mathcal{S}_{n,n}| = |\text{2143-avoiding Dumont permutations}|$
- Extend the definition of faro words to shuffles of two (or more) words avoiding a pattern different than 21.
- Investigate the distribution and avoidance of classical patterns
- Can one characterize the image by g^{-1} of classical pattern statistics on paths?

Presented at the Virtual 2020 Permutation Patterns Workshop, 30 June - 1 July.

Version 2. A typo and the definition of the order $\preceq$ are corrected.

Shuffle photo source : https://fr.m.wikipedia.org/wiki/Fichier:Riffle_shuffle.jpg