Pattern distributions in Dyck paths with a first return decomposition constrained by height

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Abstract

We provide generating functions for the popularity and the distribution of patterns of length at most three over the set of Dyck paths having a first return decomposition constrained by height.

Keywords: Dyck and Motzkin paths, pattern statistic, distribution, popularity

1 Introduction and notation

Dyck paths with a constrained first return decomposition were introduced in [4] where the authors present both enumerative results using generating functions and a constructive bijection with the set of Motzkin paths. In [5], a similar study has been conducted for Motzkin, 2-colored Motzkin, Schröder and Riordan paths. In the literature, many papers deal with the enumeration of classical Dyck paths according to different parameters, e.g. length, number of peaks, valleys, double rises and other pattern occurrences [10, 15, 16, 18, 19, 20, 21, 24]. Restricted classes of Dyck paths have also been considered, for instance Barcucci et al. [2] consider Dyck paths having a non-decreasing height sequence of valleys (see also [8, 9]). Other papers deal with Motzkin paths using similar methods [3, 6, 11, 12, 17, 22, 25]. Motzkin and Catalan numbers appear alongside in many situations [11] and several one-to-one correspondences exist between restricted Dyck paths and Motzkin paths. For instance, Dyck paths avoiding a triple rise are enumerated by the Motzkin numbers [7].

In this paper, we focus on the distribution and the popularity of patterns of length at most three in constrained Dyck paths defined in [4]. Our method consists in showing how patterns are getting transferred from constrained Dyck paths to Motzkin paths, which settles us in a more suitable ground in order to provide generating functions for the distribution and the popularity.

A *Motzkin path* of length $n \geqslant 0$ is a lattice path consisting of flat steps $F = (1, 0)$, up steps $U = (1, 1)$ and down steps $D = (1, -1)$, starting at $(0, 0)$,

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ending at $(n, 0)$ and never going below the x-axis. For $n \geqslant 0$, we denote by $\mathcal{M}_n$ the set of all Motzkin paths of length n and we set $\mathcal{M} = \bigcup_{n \geqslant 0} \mathcal{M}_n$. A Motzkin path of length $2n$ with no flat steps is a *Dyck path* of semilength n. For $n \geqslant 0$, let $\mathcal{D}_n$ be the set of all Dyck paths of semilength n and $\mathcal{D} = \bigcup_{n \geqslant 0} \mathcal{D}_n$. The cardinality of $\mathcal{D}_n$ is given by the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$, which is the general term of the sequence A000108 in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [23]. The cardinality of $\mathcal{M}_n$ is given by the nth Motzkin number $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k$ (see A000108 in [23]).

Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition [10] of the form $P = U\alpha D\beta$ where α and β are two Dyck paths in $\mathcal{D}$. In [4], the authors introduced the set $\mathcal{D}^{h, \geqslant}$ constituted of the empty Dyck path and the Dyck paths in $\mathcal{D}$ having a first return decomposition satisfying

$$h(U\alpha D) \geqslant h(\beta)$$

where $\alpha, \beta \in \mathcal{D}^{h, \geqslant}$ and h returns the maximal height of a Dyck path. For $n \geqslant 0$, let $\mathcal{D}_n^{h, \geqslant}$ be the subset of Dyck paths of semilength n in $\mathcal{D}^{h, \geqslant}$. For instance, $\mathcal{D}_3^{h, \geqslant}$ consists of four Dyck paths $UDUDUD$, $UUDDUD$, $UUDUDD$ and $UUUDDD$. In [4], the authors prove, using generating functions, that $\mathcal{D}_n^{h, \geqslant}$ and $\mathcal{M}_n$ have the same cardinality, and they present also the following bijection ϕ between these sets.

For $P \in \mathcal{D}^{h, \geqslant}$,

$$\phi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon, \\ \phi(\alpha)F & \text{if } P = \alpha UD, \\ \phi(\alpha)\phi(\gamma)U\phi(\beta)D & \text{if } P = \alpha UU\beta D\gamma D. \end{cases}$$

For instance, the images by ϕ of $UDUDUD$, $UUDDUD$, $UUDUDD$, $UUUDDD$, $UUUUDDDDUUUDDUDD$ are respectively FFF, UDF, FUD, UFD and $UUDDFUFD$. We refer to Figure 1 for an illustration of this mapping.

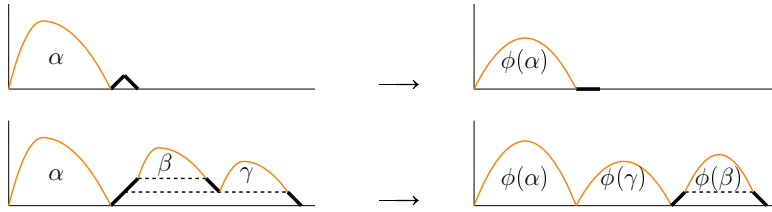Figure 1: Illustration of the bijection ϕ between $\mathcal{D}_n^{h, \geqslant}$ and $\mathcal{M}_n$.

A *statistic* on a set S of paths is an association of an integer in $\mathbb{Z}$ to each path in S. For instance the map that returns the number of steps is a statistic, and we denote by 1 (resp. 0) the constant statistic that sends any path to 1 (resp. 0). Let $\mathcal{S}$ be the set of all statistics on a set S. For X, Y $\in \mathcal{S}$, we define the statistic X + Y so that $(X + Y)(P) = X(P) + Y(P)$ for any $P \in S$, which endows $\mathcal{S}$ with a $\mathbb{Z}$-module structure. Let S and T be two sets of paths, and let $\mathcal{S}$ and $\mathcal{T}$ be the associated statistic sets. Two statistics X $\in \mathcal{S}$ and Y $\in \mathcal{T}$ have the *same distribution* if and only if there exists a bijection f from S to T

such that for any $P \in S$ we have $\mathrm{X}(P) = \mathrm{Y}(f(P))$. In this case, we say that f transports the statistic X into Y, which can be shortly written with the statistic equation $f(\mathrm{X}) = \mathrm{Y}$ (or $\mathrm{X} = \mathrm{Y}$ whenever f is the identity).

A *pattern* X of length $k \geqslant 1$ occurs in a path P if and only if P contains X as a sequence of consecutive steps. Note that other variants of pattern definition exist in the literature (see for instance [1]). From a given pattern X and a set S of paths, we associate the *statistic* X from S to $\mathbb{N}$ such that $\mathrm{X}(P)$ is the number of occurrences of the pattern X in P. The *popularity* of a pattern X in S is the total number of occurrences of the pattern X in all paths $P \in S$, that is $\sum_{P \in S} \mathrm{X}(P)$.

For instance, if $P = UUUDDUDD$ then we have $\mathrm{UD}(P) = \mathrm{UU}(P) = 2$, $\mathrm{DDD}(P) = 0$. Moreover, if $S = \{UUDD, UDUD\}$ then the popularity of the pattern UD in S is 3. Also, for any subset S of $\mathcal{M}$, we have the statistic equation $\mathrm{U} = \mathrm{D}$ since for any path in S the number of up steps equals to the number of down steps. If we restrict S to $\mathcal{D}_n$ (resp. $\mathcal{M}_n$), then we have $\mathrm{U} + \mathrm{D} = 2\mathrm{n}$ (resp. $\mathrm{U} + \mathrm{F} + \mathrm{D} = \mathrm{n}$), where n is the constant statistic $P \mapsto n$.

Considering the above bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$, the length of $\phi(P)$ is the semilength of P and we easily deduce the statistic equation:

$$\phi(\mathrm{U}) = \phi(\mathrm{D}) = \mathrm{U} + \mathrm{D} + \mathrm{F} = \mathrm{n}. \tag{1}$$

In this paper, we present statistic equations showing how $\phi\colon \mathcal{D}_n^{h,\geqslant} \to \mathcal{M}_n$ transports statistics associated to patterns of length at most three into linear combinations of other statistics. Then, this allows us to conduct our enumerative study in the more natural and simpler context of Motzkin paths, while a direct study on $\mathcal{D}_n^{h,\geqslant}$ is complicated due to the lack of an adequate recursive decomposition. So, we use statistic equations in order to derive bivariate generating functions for the distribution of patterns of length at most three, and we deduce the pattern popularity thereafter. Section 2 deals with patterns of length 2 while Section 3 deals with patterns of length 3. Many of the resulting sequences correspond to existing entries in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [23] enumerating a quantity of already known combinatorial structures, and we also obtain new sequences not yet known in [23].

2 Patterns of length 2

In this part, we provide statistic equations showing how patterns of length two behave through $\phi : \mathcal{D}_n^{h,\geqslant} \to \mathcal{M}_n$, which allows us to deduce generating functions for the distribution and the popularity of such patterns. See Table 1 for an illustration of the distributions and Table 2 for the first terms of popularity sequences.

Theorem 1. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports statistics associated to patterns of length two as follows:*

$$\phi(\mathrm{UD}) = \mathrm{F} + \mathrm{UD}, \tag{2}$$
$$\phi(\mathrm{UU}) = \phi(\mathrm{DD}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF}, \tag{3}$$
$$\phi(\mathrm{DU}) = \mathrm{FF} + \mathrm{FU} + \mathrm{DF} + \mathrm{DU}. \tag{4}$$

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Proof. For equation (2), we refer to [4].

For equation (3), we observe that for any Dyck path of semilength n an up step U is always followed by U or D, which implies the equality UU + UD = n on $\mathcal{D}_n^{h,\geqslant}$. With a similar argument and U = D, we have F+U+UD+UU+UF = n on $\mathcal{M}_n$. Using eq. (2) we obtain $\phi(\mathrm{UU}) = \mathrm{n} - \phi(\mathrm{UD}) = \mathrm{n} - (\mathrm{F} + \mathrm{UD}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF}$.

For equation (4), we observe that the statistic UU + UD + DU + DD equals $2\mathrm{n} - 1$ on $\mathcal{D}_n^{h,\geqslant}$. Using the straightforward equality UU = DD on $\mathcal{D}_n$, we obtain DU = 2n−1−2UU−UD. Applying the bijection ϕ and using eqs. (2) and (3) we obtain $\phi(\mathrm{DU}) = 2\mathrm{n} - 1 - 2(\mathrm{U} + \mathrm{UU} + \mathrm{UF}) - (\mathrm{F} + \mathrm{UD})$. On the other hand, on the set of Motzkin paths of length n we have the statistic equations U + F + D = n and FD + FU + FF + DD + DU + DF + UD + UU + UF = n − 1, which induces $2\mathrm{n}-1 = \mathrm{U}+\mathrm{F}+\mathrm{D}+\mathrm{FD}+\mathrm{FU}+\mathrm{FF}+\mathrm{DD}+\mathrm{DU}+\mathrm{DF}+\mathrm{UD}+\mathrm{UU}+\mathrm{UF}$. Also, for any Motzkin path we have U = D which implies UU + UF + UD = UD + FD + DD, and thus UU + UF = FD + DD. So, combining all these equations, we obtain $\phi(\mathrm{DU}) = \mathrm{U} + \mathrm{F} + \mathrm{D} + \mathrm{FD} + \mathrm{FU} + \mathrm{FF} + \mathrm{DD} + \mathrm{DU} + \mathrm{DF} + \mathrm{UD} + \mathrm{UU} + \mathrm{UF} - \mathrm{U} - \mathrm{D} - \mathrm{UU} - \mathrm{UF} - \mathrm{FD} - \mathrm{DD} - \mathrm{F} - \mathrm{UD} = \mathrm{FF} + \mathrm{FU} + \mathrm{DF} + \mathrm{DU}$. $\square$

Theorem 2. *The bivariate generating functions $F_p(x,y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths in $\mathcal{D}_n^{h,\geqslant}$ containing exactly k occurrences of the pattern $p \in \{UD, UU, DD, DU\}$ are*

$$F_{UD}(x,y) = \frac{x^2 - x^2 y - xy + 1 - \sqrt{-4x^2 + \left(x^2\left(y - 1\right) + xy - 1\right)^2}}{2x^2},$$

$$F_{UU}(x,y) = F_{DD}(x,y) = \frac{x^2 y^2 - x^2 y - x + 1 - \sqrt{-4x^2 y^2 + \left(x^2 y\left(y - 1\right) - x + 1\right)^2}}{2x^2 y^2},$$

$$F_{DU}(x,y) = \frac{x^2 y - x^2 - xy + 1 - \sqrt{-4x^2 + \left(x^2\left(y - 1\right) + xy - 1\right)^2}}{2x^2 y}.$$

Proof. For $p = UD$, Corollary 3 in [4] provides directly the bivariate generating function as a solution of a functional equation on Motzkin paths with respect to the number of occurrences of patterns F and UD (see eq. (2)).

For $p = UU$, there are two ways to obtain the generating function. First, we know that UU + UD = n for Dyck paths of semilength n. So, we can obtain directly the generating function by calculating $F_{UD}(xy, \frac{1}{y})$. Moreover we know that $\phi(\mathrm{UU}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF}$ (see eq. (3)). So, we decompose the set $\mathcal{M}$ of all Motzkin paths in the following way:

$$\mathcal{M} = \epsilon + F\mathcal{M} + UD\mathcal{M} + U(\mathcal{M}\backslash\epsilon)D\mathcal{M}.$$

The generating function for $\epsilon + F\mathcal{M}$ is given by $1 + xM(x,y)$; the g.f. for $UD\mathcal{M}$ is $x^2 yM(x,y)$ since UD contains one occurrence of U; the g.f. for $U(\mathcal{M}\backslash\epsilon)D\mathcal{M}$ is $x^2 y^2(M(x,y)-1)M(x,y)$ since a path in $U(\mathcal{M}\backslash\epsilon)D$ starts with U and its first two steps are either UU or UF. This induces the functional equation

$$M(x,y) = 1 + xM(x,y) + x^2 yM(x,y) + x^2 y^2(M(x,y) - 1)M(x,y),$$

which also gives the expected result.

For $p = DU$, the result is obtained by using the equality $DU = UD - 1$ and evaluating $F_{DU}(x,y) = 1 + \frac{F_{UD}(x,y) - F_{UD}(x,0)}{y}$. Note that we can obtain this result using eq. (4). $\qquad\square$

Table 1 gives the number of paths in $\mathcal{D}_n^{h,\geqslant}$ having k peaks UD for $1 \leqslant n \leqslant 11$ and $1 \leqslant k \leqslant 8$. Applying standard techniques from generating function theory, we verify that the second row corresponds to the sequence of quarter-squares numbers, $\lfloor (n^2/4) \rfloor$, which is the third row of Lozanić's triangle (see A034851 and [14]). The third row is a shift of the sequence A005994 corresponding to alkane numbers $l(7,n)$ from fifth row of Lozanić's triangle, which enumerates certain symmetries exhibited by chemical entities (alkane) consisting of hydrogen and carbon atoms arranged in a tree-like structure. Third diagonal corresponds to the squares, while fourth diagonal generates octahedral numbers, $n(2n^2 + 1)/3$ (see A005900).

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2		1	2	4	6	9	12	16	20	25	30
3			1	3	9	19	38	66	110	170	255
4				1	4	16	44	111	240	485	900
5					1	5	25	85	260	676	1615
6						1	6	36	146	526	1602
7							1	7	49	231	959
8								1	8	64	344
9									1	. . .	. . .
Σ	1	2	4	9	21	51	127	323	835	2188	5798

Table 1: Number of paths in $\mathcal{D}_n^{h,\geqslant}$ having k peaks UD, or equivalently $k - 1$ valleys DU, or equivalently $n - k$ double rises UU.

Corollary 1. *For $n \geqslant 0$, the popularity of pattern $p \in \{UU, UD, DD, DU\}$ in $\mathcal{D}_n^{h,\geqslant}$ is given by the generating function $G_p(x)$:*

$$G_{UD}(x) = \frac{(x-1)\sqrt{-3x^2 - 2x + 1} - 3x^2 - 2x + 1}{2x(3x-1)},$$

$$G_{UU}(x) = G_{DD}(x) = \frac{\sqrt{-3x^2 - 2x + 1}\left(x^2 + 2x - 2\right) + x^3 - 3x^2 - 4x + 2}{2x^2\sqrt{-3x^2 - 2x + 1}},$$

$$G_{DU}(x) = \frac{\left(x^2 - 1\right)\sqrt{-3x^2 - 2x + 1} - x^3 - 3x^2 - x + 1}{2x^2\sqrt{-3x^2 - 2x + 1}}.$$

Proof. The generating function $G_p(x)$ of popularity is directly deduced from the bivariate generating function of pattern distribution

$$G_p(x) = \left.\frac{\partial F_p(x,y)}{\partial y}\right|_{y=1}.$$

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$\square$

See Table 2 for the first terms of the popularity sequences. The popularity of the pattern UD generates a shift of the sequence A025566 in [23]. As suggested in [23], the same sequence enumerates the first differences of the directed animals sequence A005773, and also Motzkin paths of length $2n$ whose last weak valley occurs immediately after step n.

The popularity sequence for DU is the sequence A025567. As mentioned in [13] by Ferrari and Munarini, this sequence corresponds to the number of edges in Hasse diagram of Motzkin paths, where the partial order is defined by the coverings $FF \mapsto UD$, $FU \mapsto UF$, $DF \mapsto FD$, $DU \mapsto FF$. This is an immediate consequence of the fact that ϕ maps pattern DU from constrained Dyck paths to the patterns FF, FU, DF, DU in Motzkin paths.

The popularity sequence for UU (or DD) does not yet appear in [23].

Pattern	Popularity sequence	OEIS
UD	$1, 3, 8, 22, 61, 171, 483, 1373, 3923, 11257, 32418, 93644$	A025566
DU	$0, 1, 4, 13, 40, 120, 356, 1050, 3088, 9069, 26620, 78133$	A025567
UU, DD	$0, 1, 4, 14, 44, 135, 406, 1211, 3592, 10623, 31260, 92488$	

Table 2: Popularity of length two patterns in $\mathcal{D}_n^{h,\geqslant}$ for $1 \leqslant n \leqslant 12$.

3 Patterns of length 3

In this part we investigate how ϕ transports statistics associated to patterns of length three, and for each of them we provide generating functions for the distribution and popularity.

We need the following notations. For a step $X \in \{U, D, F\}$, an occurrence of the pattern X^+ inside a path P is an occurrence of X^k, where X^k consists of k consecutive repetitions of X, for some $k \geqslant 1$. The associated statistic of X^+ (denoted X^+) will be equal to $\sum_{k \geqslant 1} \mathrm{X}^k$ where X^k is the statistic giving the number of occurrences of X^k. More generally, for two possibly empty sequences of steps Y and Z we define the pattern YX^+Z as patterns of the form YX^kZ for $k \geqslant 1$, and its associated statistic as $\mathrm{YX^+Z} = \sum_{k \geqslant 1} \mathrm{YX^kZ}$. For instance, an occurrence of the pattern DU^+D can be an occurrence of DUD, $DUUD$, $DUUUD$, and so on... In the path $UUF\boldsymbol{DUU}DF\boldsymbol{DU}\boldsymbol{DUUU}DDFFDD$ we count three occurrences of the pattern DU^+D. A path contains a dotted pattern $\bullet Y$ (resp. $Y\bullet$) if and only if it starts (resp. ends) with Y, and we use the notations $\bullet\mathrm{Y}$, $\mathrm{Y}\bullet$ for the associated statistics.

Any pattern UU in a Dyck path is immediately followed by an up step or a down step, implying the statistic equation $\mathrm{UU} = \mathrm{UUU} + \mathrm{UUD}$. Also, any pattern UU in a Dyck path P is either at the beginning of P or immediately preceded by an up or a down step, so $\mathrm{UU} = \bullet\mathrm{UU} + \mathrm{UUU} + \mathrm{DUU}$. Using similar arguments we obtain the two following systems of statistic equations.

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$$(a) \begin{cases} \text{UU} = \text{UUU} + \text{UUD} \\ \text{UU} = \text{UUU} + \text{DUU} + \bullet\text{UU} \\ \text{UD} = \text{UUD} + \text{DUD} + \bullet\text{UD} \\ \text{DU} = \text{DUU} + \text{DUD} \end{cases}, \quad (b) \begin{cases} \text{DD} = \text{DDD} + \text{UDD} \\ \text{DD} = \text{DDD} + \text{DDU} + \text{DD}\bullet \\ \text{UD} = \text{UDD} + \text{UDU} + \text{UD}\bullet \\ \text{DU} = \text{DDU} + \text{UDU} \end{cases}.$$

Observe that for any $P \in \mathcal{D}_n^{h,\geqslant}$, $n \geqslant 1$, we have $\bullet\text{UD}(P) = 1$ (resp. $\bullet\text{UU}(P) = 0$) when $P = (UD)^n$, and $\bullet\text{UD}(P) = 0$ (resp. $\bullet\text{UU}(P) = 1$) otherwise. So, we have the statistic equations $\phi(\bullet\text{UD}) = \delta_{F^n}$ and $\phi(\bullet\text{UU}) = 1 - \phi(\bullet\text{UD})$, where δ_{F^n} is the Dirac statistic defined by $\delta_{F^n}(P) = 1$ whenever $P = F^n$, and 0 otherwise.

Knowing the image through ϕ of only one statistic $X \in \{\text{UUU}, \text{UUD}, \text{DUU}, \text{DUD}\}$ and using results from Section 2, we can obtain the expressions of the images of the three other statistics from the system (a). The same reasoning holds for the second system (b). So, we split Section 3 into two subsections, each dealing with one of the two systems.

3.1 The patterns UUU, UUD, DUU, DUD

Theorem 3. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UUD *as follows:*
$$\phi(\text{UUD}) = \text{UF}^+\text{D} + \text{UD}.$$

Proof. We proceed by induction on n. For $n = 1$, we have $\text{UUD}(UD) = 0$. Since $\phi(UD) = F$, we obtain $\text{UF}^+\text{D}(F) + \text{UD}(F) = 0$, and the result holds. We assume the result for $k \leqslant n$, and we will prove it for $n+1$ by distinguishing two main cases.

- Whenever $P = \alpha UD$, we have $\phi(P) = \phi(\alpha)F$, and $(\text{UF}^+\text{D} + \text{UD})(\phi(P)) = (\text{UF}^+\text{D} + \text{UD})(\phi(\alpha))$. Using induction hypothesis, $(\text{UF}^+\text{D} + \text{UD})(\phi(\alpha)) = \text{UUD}(\alpha)$ which is also equal to $\text{UUD}(\alpha UD))$, proving the first case.

- Whenever $P = \alpha UU\beta D\gamma D$, we have $\phi(P) = \phi(\alpha)\phi(\gamma)U\phi(\beta)D$, and $(\text{UF}^+\text{D}+\text{UD})(\phi(P)) = (\text{UF}^+\text{D}+\text{UD})(\phi(\alpha))+(\text{UF}^+\text{D}+\text{UD})(\phi(\gamma)U\phi(\beta)D)$. Using induction hypothesis, we obtain $(\text{UF}^+\text{D}+\text{UD})(\phi(P)) = \text{UUD}(\alpha)+(\text{UF}^+\text{D}+\text{UD})(\phi(\gamma)U\phi(\beta)D)$. Now we distinguish three subcases in order to prove that $(\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)U\phi(\beta)D) = \text{UUD}(UU\beta D\gamma D)$.

If β is the empty path, then $(\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)U\phi(\beta)D) = 1 + (\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)) = 1 + \text{UUD}(\gamma) = \text{UUD}(UUD\gamma D)$.

If $\beta = (UD)^k$ for some $k \geqslant 1$, then $(\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)U\phi((UD)^k)D) = (\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)UF^kD) = (\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)) + 1 = \text{UUD}(\gamma) + 1 = \text{UUD}(UU(UD)^k D\gamma D) = \text{UUD}(UU\beta D\gamma D)$.

If β starts with a double rise UU, then $\phi(\beta)$ contains at least one up step ($\phi(\beta) \neq F^\ell$ for any $\ell \geqslant 0$). So, we have $(\text{UF}^+\text{D} + \text{UD})(\phi(\gamma)U\phi(\beta)D) = (\text{UF}^+\text{D}+\text{UD})(\phi(\gamma)) + (\text{UF}^+\text{D}+\text{UD})(\phi(\beta)) = \text{UUD}(\gamma) + \text{UUD}(\beta)$, which equals to $\text{UUD}(UU\beta D\gamma D)$.

Considering these three cases, the second case is proved and the induction is completed. $\qquad\square$

Theorem 4. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UUU *as follows:*
$$\phi(\text{UUU}) = \text{UF}^+\text{D} + 2\big(\text{UF}^+\text{U} + \text{UU}\big).$$

Proof. Using Theorem 3 and the equation UU = UUD + UUU of system (a), we have $\phi(\text{UUU}) = \phi(\text{UU}) - \phi(\text{UUD}) = \phi(\text{UU}) - \text{UF}^+\text{D} - \text{UD}$, and using Theorem 1, we have $\phi(\text{UUU}) = \text{U} + \text{UU} + \text{UF} - \text{UF}^+\text{D} - \text{UD}$. Since we have U = UD + UU + UF on $\mathcal{M}_n$, we obtain $\phi(\text{UUU}) = 2(\text{UU} + \text{UF}) - \text{UF}^+\text{D}$. Using the trivial equation UF = $\text{UF}^+\text{D} + \text{UF}^+\text{U}$ for Motzkin paths, we complete the proof. $\square$

Theorem 5. *For $n \geqslant 1$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DUU *as follows:*

$$\phi(\text{DUU}) = \text{UF}^+\text{D} + \text{UD} + \delta_{F^n} - 1,$$

where δ_{F^n} is the Dirac statistic defined by $\delta_{F^n}(P) = 1$ whenever $P = F^n$, and 0 otherwise.

Proof. Using the difference of the first two equations of (a), we obtain DUU = UUD−•UU, and finally, $\phi(\text{DUU}) = \phi(\text{UUD}) - \phi(\bullet\text{UU}) = \text{UF}^+\text{D} + \text{UD} - \phi(\bullet\text{UU})$. As discussed at the beginning of Section 3, $\phi(\bullet\text{UU}) = 1 - \delta_{F^n}$, which completes the proof. $\square$

Theorem 6. *For $n \geqslant 1$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DUD *as follows:*

$$\phi(\text{DUD}) = \text{F} - \text{UF}^+\text{D} - \delta_{F^n}$$

Proof. We have DUD = UD − UUD − •UD, which implies that $\phi(\text{DUD}) = \phi(\text{UD}) - \phi(\text{UUD}) - \phi(\bullet\text{UD})$. From Theorem 1, Theorem 3 and the observations after the systems (a) and (b) we obtain $\phi(\text{DUD}) = \text{UD} + \text{F} - \text{UD} - \text{UF}^+\text{D} - \delta_{F^n} = \text{F} - \text{UF}^+\text{D} - \delta_{F^n}$. $\square$

Theorem 7. *The bivariate generating functions $F_p(x,y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths in $\mathcal{D}_n^{h,\geqslant}$ containing exactly k occurrences of the pattern $p \in \{UUU, UUD, DUU, DUD\}$ are given by the following expressions listed in the same order as written above.*

$$\frac{x^3y - x^3 - x^2y^2 + 2x - 1 + \sqrt{(x - xy - 1)(x^2 - xy + x - 1)(x^3 - x^3y + x^2y^2 + 2x^2y - 2xy - 2x + 1)}}{2x^2y^2(x-1)},$$

$$\frac{x^2y - 2x^2 + 2x - 1 + \sqrt{(x^2y - 1)(x^2y - 4x^2 + 4x - 1)}}{2x^2(x-1)},$$

$$\frac{2x - x^2y - 1 + \sqrt{(x^2y - 1)(x^2y - 4x^2 + 4x - 1)}}{2x^2y(x-1)},$$

$$\frac{x^3y - x^3 - x^2y^2 + 2xy - 1 + \sqrt{(xy - x - 1)(x^2 + yx - x - 1)(x^3y - x^3 + x^2y^2 + 2x^2y - 2xy - 2x + 1)}}{2x^2(xy-1)}.$$

Proof. For $p = UUU$ and using Theorem 4, we have $\phi(\text{UUU}) = \text{UF}^+\text{D} + 2(\text{UF}^+\text{U} + \text{UU})$. So, we decompose Motzkin paths according to the patterns UF^+D, UF^+U, and UU in order to exhibit a functional equation having $F_p(x,y)$ as solution:

$$\mathcal{M} = \epsilon + F\mathcal{M} + UD\mathcal{M} + U\mathcal{M}_0 D\mathcal{M} + U(\mathcal{M}\backslash(\mathcal{M}_0 \cup \epsilon))D\mathcal{M},$$

8

where $\mathcal{M}_0$ is the subset of $\mathcal{M}$ consisting of paths of the form F^k for $k \geqslant 1$. The generating function for $\epsilon + F\mathcal{M} + UD\mathcal{M}$ is $1 + xF_{UUU}(x,y) + x^2 F_{UUU}(x,y)$; the g.f. for $U\mathcal{M}_0 D\mathcal{M}$ is $x^2 y \frac{x}{1-x} F_{UUU}(x,y)$ since a path in $U\mathcal{M}_0 D$ is an occurrence of UF^+D; the g.f. for $U(\mathcal{M}\backslash(\mathcal{M}_0 \cup \epsilon))D\mathcal{M}$ is $x^2 y^2 \left(F_{UUU}(x,y) - \frac{1}{1-x} \right) F_{UUU}(x,y)$ since a path in $U(\mathcal{M}\backslash(\mathcal{M}_0 \cup \epsilon))D$ starts with either an occurrence of UU or an occurrence of UF^+U.

So the functional equation is:

$$F_{UUU}(x,y) = 1 + xF_{UUU}(x,y) + x^2 F_{UUU}(x,y) + x^2 y \frac{x}{1-x} F_{UUU}(x,y) +$$

$$+ x^2 y^2 \left(F_{UUU}(x,y) - \frac{1}{1-x} \right) F_{UUU}(x,y).$$

A simple calculation (with Maple for instance) provides the result.

All other generating functions are obtained using a similar method, so we do not give the proofs here. □

The generating function $G_p(x)$ for the popularity of the pattern $p \in \{UUU, UUD, DUU, DUD\}$ is obtained directly by evaluating $\frac{\partial F_p(x,y)}{\partial y}\big|_{y=1}$. Table 3 provides the first terms of the associated sequences. See also Table 4 for an illustration of the distribution of $p \in \{UUU, UUD, DUU, DUD\}$.

Pattern	Popularity sequence	OEIS
UUU	$0, 0, 1, 5, 19, 65, 210, 658, 2023, 6147, 18534, 55594$	
UUD	$0, 1, 3, 9, 25, 70, 196, 553, 1569, 4476, 12826, 36894$	A097861
DUU	$0, 0, 0, 1, 5, 20, 70, 231, 735, 2289, 7029, 21384$	A304011 ?
DUD	$0, 1, 4, 12, 35, 100, 286, 819, 2353, 6780, 19591, 56749$	

Table 3: Popularity of $p \in \{UUU, UUD, DUU, DUD\}$ in $\mathcal{D}_n^{h,\geqslant}$ for $1 \leqslant n \leqslant 12$.

The popularity sequence for UUD in $\mathcal{D}_n^{h,\geqslant}$ is the sequence A097861, corresponding to the number of humps in all Motzkin paths of length n (a hump equals UF^+D or UD in our notation). The popularity for DUU seems to generate the sequence A304011, but we did not succeed in proving this fact, so we leave it as a conjecture.

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	2	3	5	8	13	21	34	55
1			1	3	8	18	38	76	147
2				1	4	14	40	104	250
3					1	5	21	71	215
4						1	6	30	119
5							1	7	40

(a) UUU

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	1	2	3	6	10	20	36
1		1	2	3	7	13	30	58	130
2			1	3	6	16	35	91	199
3				1	4	10	30	75	216
4					1	5	15	50	140
5						1	6	21	77

(b) DUD

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1		1	3	7	15	31	63	127	255
2				1	5	18	56	160	432
3						1	7	34	138
4								1	9

(c) UUD

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	64	128	256
1				1	5	18	56	160	432
2						1	7	34	138
3								1	9

(d) DUU

Table 4: Number of paths from $\mathcal{D}_n^{h,\geqslant}$ having k occurrences of the considered pattern.

Dyck paths from $\mathcal{D}_n^{h,\geqslant}$ avoiding UUU, DUU respectively generate Fibonacci numbers and integer squares. Those avoiding DUD seem to correspond to A007562 (number of planted trees where non-root, non-leaf nodes at even distance from root are of degree 2). At the present time, there is no closed form for the generating function of sequence A007562. Note that any path avoiding DUU has at most one occurrence of UUD. Also, Dyck paths containing two occurrences of UUD in $\mathcal{D}_n^{h,\geqslant}$ generate a shift of sequence A001793, which corresponds to a subsequence in the triangle of coefficients of Chebyshev's polynomials which is sequence A053120.

3.2 The patterns DDD, DDU, UDD, UDU

Theorem 8. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UDU *as follows:*
$$\phi(\mathrm{UDU}) = \mathrm{FF} + \mathrm{FUD}$$

Proof. We proceed by induction on n. For $n = 1$, we have $\mathrm{UDU}(UD) = 0$. With $\phi(UD) = F$, we obtain $\mathrm{FF}(F) + \mathrm{FUD}(F) = 0$, and the result holds. We assume the result for $k \leqslant n$, and we will prove it for $n + 1$.

- Whenever $P = \alpha UD$, we have $\phi(P) = \phi(\alpha)F$, and $(\mathrm{FF} + \mathrm{FUD})(\phi(P)) = \mathrm{FF}(\phi(\alpha)F) + \mathrm{FUD}(\phi(\alpha)F) = \mathrm{FF}(\phi(\alpha)F) + \mathrm{FUD}(\phi(\alpha))$. We distinguish two cases: (i) $\phi(\alpha)$ ends with F, and (ii) otherwise. In the case (i), α ends with UD, and thus $(\mathrm{FF} + \mathrm{FUD})(\phi(P)) = 1 + \mathrm{FF}(\phi(\alpha)) + \mathrm{FUD}(\phi(\alpha))$. Using the induction hypothesis we have $(\mathrm{FF} + \mathrm{FUD})(\phi(P)) = 1 + \mathrm{UDU}(\alpha) = \mathrm{UDU}(\alpha UD) = \mathrm{UDU}(P)$. In the case (ii), α does not end with UD, and thus $\phi(\alpha)$ does not end

with F. So, we have $(\mathrm{FF}+\mathrm{FUD})(\phi(P)) = \mathrm{FF}(\phi(\alpha))+\mathrm{FUD}(\phi(\alpha))$, and using the induction hypothesis $(\mathrm{FF}+\mathrm{FUD})(\phi(P)) = \mathrm{UDU}(\alpha) = \mathrm{UDU}(\alpha UD) = \mathrm{UDU}(P)$.

- Whenever P does not end with UD, we have $P = \alpha UU\beta D\gamma D$, and $(\mathrm{FF} + \mathrm{FUD})(\phi(P)) = (\mathrm{FF} + \mathrm{FUD})(\phi(\alpha)\phi(\gamma)U\phi(\beta)D)$. Note that α cannot end with UD, otherwise it would contradict $P \in \mathcal{D}_n^{h,\geqslant}$. This means that $\phi(\alpha)$ cannot end with F. So, all the possible occurrences of FF in $\phi(P)$ belong necessarily to $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$, which implies that $\mathrm{FF}(\phi(P)) = \mathrm{FF}(\phi(\alpha)) + \mathrm{FF}(\phi(\beta)) + \mathrm{FF}(\phi(\gamma))$. On the other hand, the possible occurrences of FUD in $\phi(P)$ belong necessarily to $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$, and eventually at the junction of $\phi(\gamma)$ and $U\phi(\beta)D$ whenever $\phi(\gamma)$ ends with F and $\phi(\beta) = \epsilon$. So, we distinguish two cases: (a) $\phi(\gamma)$ ends with F and $\beta = \epsilon$, and (b) otherwise. In the case (a), we have $\mathrm{FUD}(\phi(P)) = 1 + \mathrm{FF}(\phi(\alpha)) + \mathrm{FF}(\phi(\beta)) + \mathrm{FF}(\phi(\gamma)) + \mathrm{FUD}(\phi(\alpha)) + \mathrm{FUD}(\phi(\beta)) + \mathrm{FUD}(\phi(\gamma))$, and using the induction hypothesis, we obtain $\mathrm{FUD}(\phi(P)) = 1 + \mathrm{UDU}(\alpha) + \mathrm{UDU}(\beta) + \mathrm{UDU}(\gamma) = \mathrm{UDU}(\alpha UU\beta D\gamma D) = \mathrm{UDU}(P)$. In the case (b), we have $\mathrm{FUD}(\phi(P)) = \mathrm{FF}(\phi(\alpha)) + \mathrm{FF}(\phi(\beta)) + \mathrm{FF}(\phi(\gamma)) + \mathrm{FUD}(\phi(\alpha)) + \mathrm{FUD}(\phi(\beta)) + \mathrm{FUD}(\phi(\gamma))$, and using the induction hypothesis, we obtain $\mathrm{FUD}(\phi(P)) = \mathrm{UDU}(\alpha) + \mathrm{UDU}(\beta) + \mathrm{UDU}(\gamma) = \mathrm{UDU}(\alpha UU\beta D\gamma D) = \mathrm{UDU}(P)$. So, the proof is complete. $\qquad\square$

Theorem 9. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UDD *as follows:*

$$\phi(\mathrm{UDD}) = \mathrm{FD} + \mathrm{UD} + \mathrm{FUU} + \mathrm{FUF}.$$

Proof. Considering the third equation of system (b), we have $\phi(\mathrm{UDD}) = \phi(\mathrm{UD}) - \phi(\mathrm{UDU}) - \phi(\mathrm{UD}\bullet)$. Using Theorems 1 and 8, we obtain $\phi(\mathrm{UDD}) = \mathrm{F} + \mathrm{UD} - \mathrm{FF} - \mathrm{FUD} - \phi(\mathrm{UD}\bullet)$. In any Motzkin path P, a flat step is either at the end of P, or followed by F, or D, or UU, or UD, or UF, that is $\mathrm{F} = \mathrm{F} \bullet + \mathrm{FF} + \mathrm{FD} + \mathrm{FUU} + \mathrm{FUD} + \mathrm{FUF}$. Then, we obtain $\phi(\mathrm{UDD}) = \mathrm{F}\bullet + \mathrm{UD} + \mathrm{FD} + \mathrm{FUU} + \mathrm{FUF} - \phi(\mathrm{UD}\bullet)$. Using the definition of ϕ in Introduction, it is clear that ϕ transports $\mathrm{UD}\bullet$ into $\mathrm{F}\bullet$, and the result holds. $\qquad\square$

Theorem 10. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DDU *as follows:*

$$\phi(\mathrm{DDU}) = \mathrm{DF} + \mathrm{DU} + \mathrm{FUU} + \mathrm{FUF}.$$

Proof. Combining the fourth equation $\mathrm{DU} = \mathrm{DDU} + \mathrm{UDU}$ of system (b) with Theorems 1 and 8, we obtain $\phi(\mathrm{DDU}) = \mathrm{FF} + \mathrm{FU} + \mathrm{DF} + \mathrm{DU} - \mathrm{FF} - \mathrm{FUD} = \mathrm{FU} + \mathrm{DF} + \mathrm{DU} - \mathrm{FUD}$. The proof is completed using the straightforward equation $\mathrm{FU} = \mathrm{FUU} + \mathrm{FUF} + \mathrm{FUD}$ on Motzkin paths. $\qquad\square$

Theorem 11. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DDD *as follows:*

$$\phi(\mathrm{DDD}) = 2(\mathrm{UU} + \mathrm{UF}) - \mathrm{FD} - \mathrm{FUU} - \mathrm{FUF}.$$

Proof. Combining the first equation DD = DDD + UDD of system (*b*) with Theorems 1 and 9, we obtain $\phi(\mathrm{DDD}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF} - \mathrm{FD} - \mathrm{UD} - \mathrm{FUU} - \mathrm{FUF}$. Using $\mathrm{U} = \mathrm{UF} + \mathrm{UD} + \mathrm{UU}$ on Motzkin paths, the result holds. $\qquad\square$

Theorem 12. *The bivariate generating functions $F_p(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths in $\mathcal{D}_n^{h,\geqslant}$ containing exactly k occurrences of the pattern $p \in \{UDU, UDD, DDU, DDD\}$ are given by the following expressions listed in the same order as written above.*

$$\frac{1 + x(x^2 - x^2 y - y) - \sqrt{A}}{2x^2\,(x - xy + 1)},$$

$$\frac{1 + x(x^2 y - x^2 - xy + x - 1) - \sqrt{B}}{2x^2\,(xy - x + 1)^2},$$

$$\frac{1 + x(2x^2 y^2 - 3x^2 y + x^2 + xy - x - 1) - \sqrt{C}}{2x^2 y\,(xy - x + 1)},$$

$$\frac{1 - x\left(x^2 y^2 - x^2 y - xy^2 + x + 1\right) - \sqrt{D}}{2x^2\,(xy - x - y)^2}$$

where

$$A = (x + 1)\left(x^2 y - x^2 + xy - x - 1\right)\left(x^3 y - x^3 - 2x^2 y + 2x^2 + xy + 2x - 1\right),$$

$$B = (x + 1)\left(x^2 y - x^2 + 1\right)\left(x^3 y - x^3 - 3x^2 y + 3x^2 - 3x + 1\right),$$

$$C = (x + 1)\left(x^2 y - x^2 + 1\right)\left(x^3 y - x^3 - 3x^2 y + 3x^2 - 3x + 1\right),$$

$$D = (xy + 1)\left(x^2 y - x^2 - xy + x - 1\right)\left(x^3 y^2 - x^3 y - x^2 y^2 - 2x^2 y + 3x^2 + 2xy + x - 1\right).$$

Proof. Let $A_p(x, y)$ (resp. $B_p(x, y)$) be the bivariate generating function where the coefficient of $x^n y^k$ is the number of Motzkin paths of length n ending with a flat step (resp. down step) having exactly k occurrences of the patterns related to the statistic $\phi(\mathrm{p})$ obtained in the r.h.s. of equations of Theorems 8,9,10,11. Using a refinement of the classical decompositions of non-empty Motzkin paths by taking into account the occurrences of the considered patterns, we deduce functional equations for $A_p(x, y)$ and $B_p(x, y)$. The method being classic, we do not give any more details. The solutions are obtained by a simple calculation.

For $p = UDU$:
$$\begin{cases} F_p(x, y) & = 1 + A_p(x, y) + B_p(x, y) \\ A_p(x, y) & = x + xy A_p(x, y) + x B_p(x, y) \\ B_p(x, y) & = x^2 + x^2 y A_p(x, y) + x^2 B_p(x, y) + x^2 F_p(x, y)(F_p(x, y) - 1); \end{cases}$$

For $p = UDD$:
$$\begin{cases} F_p(x, y) = 1 + A_p(x, y) + B_p(x, y) \\ A_p(x, y) = x F_p(x, y) \\ B_p(x, y) = x^2 y F_p(x, y) + x^2 y A_p(x, y) + x^2 B_p(x, y) + \\ \qquad + x^2 B_p(x, y)^2 + x^2 y A_p(x, y) B_p(x, y) + x^2 y^2 A_p(x, y)^2 + x^2 y A_p(x, y) B_p(x, y); \end{cases}$$

For $p = DDU$:

$$\begin{cases} F_p(x,y) = 1 + A_p(x,y) + B_p(x,y) \\ A_p(x,y) = x + xA_p(x,y) + xyB_p(x,y) \\ B_p(x,y) = x^2F_p(x,y) + x^2yA_p(x,y)(F_p(x,y)-1) + x^2A_p(x,y)+ \\ \qquad + x^2yB_p(x,y)F_p(x,y); \end{cases}$$

For $p = DDD$:

$$\begin{cases} F_p(x,y) = 1 + A_p(x,y) + B_p(x,y) \\ A_p(x,y) = xF_p(x,y) \\ B_p(x,y) = x^2F_p(x,y) + x^2y^2B_p(x,y) + x^2y^2A_p(x,y)/y + x^2y^2A_p(x,y)^2/y^2+ \\ \qquad + x^2y^2A_p(x,y)B_p(x,y)/y + x^2y^2B_p(x,y)^2 + x^2y^2A_p(x,y)B_p(x,y)/y. \end{cases}$$

The above equations are intentionally left in non-simplified forms, in order to allow the reader to easily retrieve the refined decompositions from the classical decomposition of Motzkin paths $\mathcal{M} = \epsilon + \mathcal{M}F + \mathcal{M}U\mathcal{M}D$. $\square$

Generating function $G_p(x)$ for the popularity of a pattern p is obtained directly by evaluating $\frac{\partial F_p(x,y)}{\partial y}\big|_{y=1}$. Table 6 provides the first terms of the generated sequences. See also Table 5 for an illustration of the distribution of $p \in \{UDU, UDD, DDU, DDD\}$.

Unlike what happens for classical Dyck paths, the popularity of UDU in $\mathcal{D}_{n+1}^{h,\geqslant}$ is equal to the popularity of UD in $\mathcal{D}_n^{h,\geqslant}$, while the corresponding distributions are different. Dyck paths from $\mathcal{D}_n^{h,\geqslant}$ avoiding a pattern UDU (resp. DDD) are counted by the Generalized Catalan numbers (resp. by the numbers of ordered trees with n edges and having no branches of length 1), which corresponds to the sequence A004148 (resp. A026418).

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	2	3	6	11	22	43	87	176
1			1	2	7	16	43	102	251
2				1	2	10	25	80	208
3					1	2	13	34	130
4						1	2	17	46
5							1	2	21

(a) DDD

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	2	4	8	17	37	82	185
1		1	1	3	7	17	41	102	252
2			1	1	4	10	28	73	200
3				1	1	5	13	41	113
4					1	1	6	16	56
5						1	1	7	19

(b) UDU

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
1			1	5	14	31	59	102	164
2					2	14	57	174	444
3							4	39	209
4									9

(c) DDU

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1		1	3	6	10	15	21	28	36
2				2	10	31	75	156	292
3						4	30	129	417
4								9	89

(d) UDD

Table 5: Number of paths from $\mathcal{D}_n^{h,\geqslant}$ having k occurences of the considered pattern.

Pattern	Popularity sequence	OEIS
DDD	$0, 0, 1, 4, 14, 46, 145, 448, 1365, 4124, 12387, 37060$	
DDU	$0, 0, 1, 5, 18, 59, 185, 567, 1715, 5146, 15363, 45715$	
UDD	$0, 1, 3, 10, 30, 89, 261, 763, 2227, 6499, 18973, 55428$	
UDU	$0, 1, 3, 8, 22, 61, 171, 483, 1373, 3923, 11257, 32418$	A025566

Table 6: Popularity of 3-length patterns in $\mathcal{D}_n^{h,\geqslant}$ for $1 \leqslant n \leqslant 12$.

4   Acknowledgements

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