MOTZKIN PATHS WITH A RESTRICTED FIRST RETURN DECOMPOSITION

Jean-Luc Baril[1]
LIB, Univ. Franche-Comté, France
barjl@u-bourgogne.fr

Sergey Kirgizov
LIB, Univ. Franche-Comté, France
kerzolster@gmail.com

Armen Petrossian
LIMOS, CNRS, Univ. Clermont Auvergne, France
armpetrossian@gmail.com

Received: 12/13/18, Accepted: 8/22/19, Published: 9/4/19

Abstract

Recently, the authors introduced new families of Dyck paths having a first decomposition constrained by the height or by the number of returns. In this work we extend the study to Motzkin paths and 2-colored Motzkin paths. For these new sets, we provide enumerative results by giving bivariate generating functions with respect to the length and another parameter, and we construct one-to-one correspondences with several restricted classes of ordered trees. We also deal with Schröder and Riordan paths. As a byproduct, we present a bijective proof of $M(x)^2 = \frac{1}{1-2x}M(\frac{x^2}{1-2x})$, where $M(x)$ is the generating function of Motzkin numbers.

1. Introduction and Notations

A *Motzkin path* of length $n \geq 0$ is a lattice path starting at point $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$, down steps $D = (1,-1)$ and flat steps $F = (1,0)$. Let $\mathcal{M}_n$ be the set of Motzkin paths of length n, and $\mathcal{M} = \bigcup_{n\geq 0}\mathcal{M}_n$. A *Dyck path* of semilength n is a Motzkin path of length $2n$ with no flat steps. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, and $\mathcal{D} = \bigcup_{n\geq 0}\mathcal{D}_n$. Dyck (resp. Motzkin) paths of semilength (resp. length) n are enumerated by the n-th Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ (resp. by the n-th Motzkin number $\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\cdot c_k$) which is the general term of the sequence A000108 (resp. A001006) in the On-line Encyclopedia of Integer Sequences [20].

[1]Corresponding author

Many studies on Motzkin and Dyck paths appear in the literature. Generally, they consist in enumerating these paths according to several parameters, such as length, height, number of occurrences of a pattern, number of returns to the x-axis (see for instance [10, 16, 17, 19] for Dyck paths and [2, 5, 7, 14] for Motzkin paths). Also, many bijections have been found between these paths and various combinatorial objects such as Young tableaux, pattern avoiding permutations, RNA shapes and so on. See [21] for an overview.

Recently in [3], the authors introduced and enumerated the subset $\mathcal{D}^{s,\diamond}$ of Dyck paths having a restricted first return decomposition. More precisely, given a function $s : \mathcal{D} \to \mathbb{N}$, called *statistic*, and a comparison operator $\diamond$ on $\mathbb{N}$, the set $\mathcal{D}^{s,\diamond}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{s,\diamond}, \\ s(U\alpha D) \diamond s(\beta). \end{cases} \tag{1}$$

For $n \geq 0$, we denote by $\mathcal{D}_n^{s,\diamond}$ the set of Dyck paths of semilength n in $\mathcal{D}^{s,\diamond}$.

Whenever $\diamond$ equals $\geq$ and $s(P)$ is the maximal height $h(P)$ reached by P (resp. $s(P)$ is the number of returns in P), they prove algebraically and bijectively that $\mathcal{D}_n^{s,\diamond}$ is in one-to-one correspondence with the set $\mathcal{M}_n$ of Motzkin paths of length n. So, its generating function is

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

and the first coefficients of x^n, $n \geq 0$, in the Taylor expansion are $1, 1, 2, 4, 9, 21, 51, 127$ (see Motzkin sequence `A001006` in [20]).

The purpose of the present paper is to extend the study to Motzkin paths that naturally generalize Dyck paths. Let $M \in \mathcal{M}$ be a non-empty Motzkin path, it can be uniquely written either $M = U\alpha D\beta$ or $M = F\alpha$ with $\alpha, \beta \in \mathcal{M}$. This decomposition will be called the *first return decomposition* of M (see Figure 1 for an illustration of this decomposition).

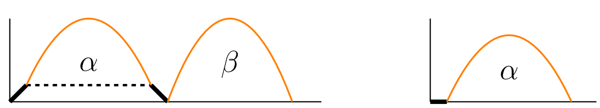

Figure 1: First return decompositions $U\alpha D\beta$ and $F\alpha$ of a Motzkin path.

Based on this decomposition and in the same way as for Dyck paths in [3], we construct a new collection of subsets of $\mathcal{M}$ as follows. Given a statistic function $s : \mathcal{M} \to \mathbb{N}$, and a comparison operator $\diamond$ on $\mathbb{N}$, the set $\mathcal{M}^{s,\diamond}$ is the union of the empty Motzkin path with all Motzkin paths M having a first return decomposition satisfying one of the two following conditions:

(C_1) $M = U\alpha D\beta$ with $\alpha, \beta \in \mathcal{M}^{s,\diamond}$ and $s(U\alpha D) \diamond s(\beta)$, or

(C_2) $M = F\alpha$ with $\alpha \in \mathcal{M}^{s,\diamond}$ and $s(F) \diamond s(\alpha)$.

For $n \geq 0$, we denote by $\mathcal{M}_n^{s,\diamond}$ the set of Motzkin paths of length n in $\mathcal{M}^{s,\diamond}$. For instance, if the operator $\diamond$ is $=$ and s is a constant statistic (*i.e.*, $s(M) = 0$ for any $M \in \mathcal{M}$), then we obviously have $\mathcal{M}_n^{s,\diamond} = \mathcal{M}_n$ for $n \geq 0$.

In this paper, we focus on the sets $\mathcal{M}^{s,\diamond}$, where the statistic $s(P)$ is either the number $r(P)$ of returns (a return is a down step D that touches the x-axis) or the maximal height $h(P)$ reached by the path.

Due to the above definition of $\mathcal{M}^{s,\diamond}$ whenever $s \in \{r, h\}$, a Motzkin path in $\mathcal{M}^{s,\geq}$ cannot contain any occurrence of FU because (C_2) implies that $0 = s(F) \geq s(\alpha)$, and thus $\alpha = F^k$ for some $k \geq 0$. Conversely, Motzkin paths in $\mathcal{M}^{s,\geq}$ can be constructed from Dyck paths in $\mathcal{D}^{s,\geq}$ by possibly adding flat steps before down steps or at the end. Since the two sets $\mathcal{D}_n^{r,\geq}$ and $\mathcal{D}_n^{h,\geq}$ are in bijection with the set $\mathcal{M}_n$ (see [3]), the generating function $M_s(x)$ of $\mathcal{M}^{s,\geq}$ for $s \in \{h, r\}$ satisfies

$$M_s(x) = \frac{1}{1-x} \cdot M\left(\frac{x^2}{1-x}\right).$$

The first coefficients of the Taylor expansion are $1, 1, 2, 3, 6, 11, 22, 43, 87, 176, 362, 748$, which correspond to the sequence `A026418` in [20].

In the same way, we define the set $\mathcal{R}^{s,\diamond}$ (resp. $\mathcal{S}^{s,\diamond}$) of Riordan paths (resp. of Schröder paths), *i.e.*, the set of Motzkin paths in $\mathcal{M}^{s,\diamond}$ with no flat steps on the x-axis (resp. where any maximal run of flats is of even length). As above, whenever $s \in \{h, r\}$, the generating functions $R_s(x)$ and $S_s(x)$ of $\mathcal{R}^{s,\diamond}$ and $\mathcal{S}^{s,\diamond}$ satisfy respectively

$$R_s(x) = M\left(\frac{x^2}{1-x}\right) \qquad \text{and} \qquad S_s(x) = \frac{1}{1-x^2} \cdot M\left(\frac{x^2}{1-x^2}\right).$$

The first coefficients of the Taylor expansion of $R_s(x)$ and $S_s(x)$ are respectively $1, 1, 1, 3, 5, 11, 21, 44, 89, 186$ and $1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132$, and the even ranks of this last sequence generate the Catalan numbers (`A097331` in [20]). Note that $S_s(x)$ is the binomial transform [18] of the Motzkin generating function, evaluated on the monomial x^2. So, we have $S_s(x) = C(x^2)$, where $C(x)$ is the generating function for the Catalan numbers [13].

The paper is organized as follows. In Section 2, we deal with the case where $s = r$ and $\diamond$ is $\geq$. We provide enumerating results for the cardinality of the set $\mathcal{M}^{r,\geq}$ according to the length and the number of returns. We give a constructive bijection ϕ between this set and the set of ordered trees with no branches of length one, and we show how ϕ transports several statistics. As a byproduct, we treat the case of Riordan and Schröder paths.

In Section 3, we focus on the set $\mathcal{M}^{h,\geq}$ (the operator $\diamond$ is $\geq$, and the statistic is the height). We provide a closed form of the generating function for the set of Motzkin paths having a given height $k \geq 0$ in $\mathcal{M}^{h,\geq}$, and we deduce a continued fraction for the generating function of $\mathcal{M}^{h,\geq}$. We give a constructive one-to-one correspondence ψ between Motzkin paths in $\mathcal{M}^{h,\geq}$ and ordered trees with no branches of length one, and we show how ψ transports several parameters. So, we deduce a constuctive bijection $\psi^{-1} \circ \phi$ from $\mathcal{M}^{r,\geq}$ to $\mathcal{M}^{h,\geq}$.

In Section 4, we extend our study to the set $\bar{\mathcal{M}}$ of 2-Motzkin paths, *i.e.*, Motzkin paths where flat steps are of two kinds: straight and wavy. Using similar reasonings already done in Sections 2 and 3, we give enumerative results for the set $\bar{\mathcal{M}}_n^{s,\geq}$, $s \in \{r, h\}$, and we present a constructive bijection between $\bar{\mathcal{M}}^{s,\geq}$ and the set of ordered unary-binary trees where the root has exactly two children. As a byproduct, we obtain a bijective proof of the equality $M(x)^2 = \frac{1}{1-2x} M(\frac{x^2}{1-2x})$.

Finally, we conclude by presenting possible extensions of this work.

2. Motzkin Paths Constrained by the Number of Returns

In this section, the statistic s is the number of down steps D that touch the x-axis (called number of returns), and the comparison operator $\diamond$ is $\geq$.

Let $A(x, y) = \sum_{k,n\geq 0} a_{n,k} x^n y^k$ be the generating function where the coefficient $a_{n,k}$ of $x^n y^k$ is the number of Motzkin paths with k returns in $\mathcal{M}_n^{r,\geq}$.

Theorem 2.1. *The generating function $A(x, y)$ is given by*

$$A(x, y) = A_0(x) + A_1(x)y + A_2(x)y^2$$

where

$$A_0(x) = \frac{1}{1-x}, \qquad A_1(x) = \frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}{2x^2(1-x)}, \text{ and}$$

$$A_2(x) = \frac{1 - 2x - x^2 + 2x^3 - x^4 - (1 - x - x^2)\sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}{2x^4(1-x)}.$$

Proof. A Motzkin path M in $\mathcal{M}_n^{r,\geq}$ is either of the form $M = F^n$, or $M = U\alpha D\beta$, where $\alpha, \beta \in \mathcal{M}^{r,\geq}$ and $r(U\alpha D) = 1 \geq r(\beta)$. Using this last inequality, β is necessarily of one of two following forms: (i) $\beta = F^k$, or (ii) $\beta = U\gamma DF^k$ with $\gamma \in \mathcal{M}^{r,\geq}$ and $k \geq 0$. Then, we obtain the functional equation

$$A(x, y) = \frac{1}{1-x} + \frac{x^2 y}{1-x} A(x, 1) + \frac{x^4 y^2}{1-x} A(x, 1)^2.$$

A straightforward calculation provides the results. □

Notice that we retrieve the above generating function $M_r(x)$ for the sets $\mathcal{M}_n^{r,\geq}$, $n \geq 0$, by calculating $A(x,1)$:

$$M_r(x) = A(x,1) = \frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 + 2x^3 - 3\,x^4}}{2x^4}.$$

The first terms of the Taylor expansion are $1 + x + 2x^2 + 3x^3 + 6x^4 + 11x^5 + 22x^6 + 43x^7 + 87x^8 + 176x^9 + 362x^{10}$ (see A026418 in [20]). Table 1 presents the first values of coefficient $a_{n,k}$.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1		1	2	4	7	13	24	46	89	176
2				1	3	8	18	40	86	185
$\sum$	1	2	3	6	11	22	43	87	176	362

Table 1: Number $a_{n,k}$ of Motzkin paths with k returns in $\mathcal{M}_n^{r,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 2$. For $k \geq 3$ and $n \geq 1$, we have $a_{n,k} = 0$.

The end of this section is dedicated to establishing a bijective link between $\mathcal{M}_n^{r,\geq}$ and a restricted set of ordered trees with n edges.

An *ordered tree* is a rooted tree where the order of subtrees matters. A *leaf* is a vertex with no child. According to the terminology used in [11], a vertex which is not a leaf or the root is called a *node*. A *branch node* is a node with at least two children, and a *branch* is a path where the extremities are either the root or a leaf or a branch node, and such that the other vertices are not branch nodes. The *length* of a branch is the number of its vertices minus one (or equivalently, the number of its edges). For $n \geq 0$, let $\mathcal{T}_n$ be the set of ordered trees with n edges, and let $\mathcal{T}_n^\star$ be the set of ordered trees with n edges and having no branches of length one. Any non-empty ordered tree $T \in \mathcal{T}_n$ can be decomposed in the form (L, R), where R consists of the root of T connected with its rightmost subtree, and L is the root of T connected with the remaining subtrees. In the following, L (resp. R) will be called the *left part* (resp. *right part*) of T. Note that the rightmost subtree of T is obtained from the right part R of T by deleting its root. Also, T is obtained from L and R by merging the roots of L and R (see Figure 2). According to this decomposition, we obtain directly the functional equation $T(x) = 1 + xT(x)^2$, where $T(x)$ is the generating function for the number of ordered trees with respect to the number of edges. As expected, the number of ordered trees with n edges is given by the n-th Catalan number.

By convenience, we adopt the following notation for an ordered tree T. We use the above decomposition $T = (L, R)$, and in the case where L is only the root, we

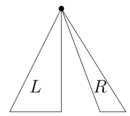

Figure 2: Decomposition (L, R) of T, where R is the root of T connected with its rightmost subtree, and L is the root of T connected with the remaining subtrees.

simplify this notation by writting $T = eR'$, where R' is the unique subtree of T (e refers to the edge between the root and the root of R'). For instance, e (resp. ee) is the root connected to a leaf (resp. the root connected to a node in turn connected to a leaf).

Now, we recursively define a map ϕ from $\mathcal{M}_n^{r,\geq}$ to the set $\mathcal{T}_{n+2}^{\star}$ for $n \geq 0$. Let M be a Motzkin path in $\mathcal{M}_n^{r,\geq}$, and $\alpha, \beta \in \mathcal{M}^{r,\geq}$:

(i) if $M = \epsilon$, then $\phi(M) = ee$,

(ii) if $M = \alpha F$ with $\phi(\alpha) = (L, R)$, then $\phi(M) = (L, eR)$,

(iii) if $M = U\alpha D$, then $\phi(M) = (\phi(\alpha), ee)$,

(iv) if $M = U\alpha DU\beta D$ with $\phi(\alpha) = (L, R)$, then $\phi(M) = (L, ee(R, \phi(\beta)))$.

An illustration of this map is given in Figure 3, and see Figure 5 for an example.

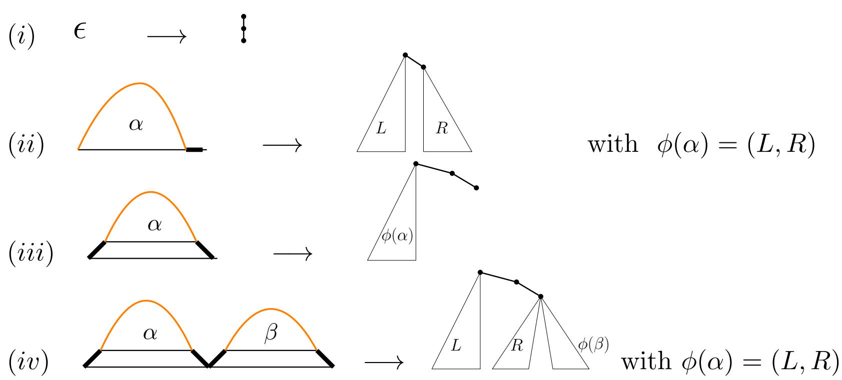

Figure 3: An illustration of the bijection ϕ.

Theorem 2.2. *For $n \geq 0$, the map ϕ is a bijection from $\mathcal{M}_n^{r,\geq}$ to $\mathcal{T}_{n+2}^{\star}$ satisfying $r(M) = \delta(\phi(M))$, where $\delta(T) = 0$ if T is a branch, and in other cases $\delta(T) = 1$ if the right part in the decomposition of T is a branch, and $\delta(T) = 2$ otherwise.*

Proof. We proceed by induction on n. Obviously, for $n = 1$, we have $\phi(F) = eee$ and $r(F) = 0 = \delta(eee)$. For $k \leq n$, we assume that ϕ is a bijection from $\mathcal{M}_k^{r,\geq}$ to $\mathcal{T}_{k+2}^{\star}$ such that $r(M) = \delta(\phi(M))$ for any $M \in \mathcal{M}_k^{r,\geq}$ and we prove the result for $n+1$. Since $\mathcal{M}_{n+1}^{r,\geq}$ and $\mathcal{T}_{n+3}^{\star}$ have the same cardinality (see [20]), it suffices to prove that if $M, M' \in \mathcal{M}_{n+1}^{r,\geq}$ with $\phi(M) = \phi(M')$ then $M = M'$. By a simple observation of Figure 3, the image by ϕ of paths satisfying (iii) are the trees in $\mathcal{T}_{n+2}^{\star}$ with a branch of length two as right part. Paths satisfying (ii) are sent on trees having the right part that starts with a branch of length at least three. Path satisfying (iv) are sent on trees having the right part that starts with a branch of length two and that contains a branch node. So, $\phi(M) = \phi(M')$ implies that M and M' belong to the same case (i), (ii), (iii), or (iv), and the recurrence hypothesis induces $M = M'$. Moreover, if $M = F^{n+1}$, then $\phi(M) = e^{n+3}$ and $r(M) = 0 = \delta(\phi(M))$; if $M = U\alpha D$, then $\phi(M) = (\phi(\alpha), ee)$ and $r(M) = 1 = \delta(\phi(M))$; if $M = U\alpha DU\beta D$ then $\phi(M) = (L, ee(R, \phi(\beta)))$ and $r(M) = 2 = \delta(\phi(M))$ which completes the proof. □

The bijection ϕ establishes correspondences between several statistics on Motzkin paths and ordered trees. For instance, the number of flats is translated into the difference between the number of edges and two times the number of branches. The number of up steps equals the number of branches minus one. The number of DU equals the number of branch nodes. Table 2 summarizes the main correspondences. We do not give formal proofs since they do not present any particular difficulties.

Corollary 2.3. *The restriction of ϕ to $\mathcal{D}_n^{r,\geq}$ establishes a one-to-one correspondence with ordered trees having $n+1$ branches in $\mathcal{T}_{2n+2}^{\star}$, which in turn are in bijection with ordered trees with $n+1$ edges.*

Corollary 2.4. *The restriction of ϕ to $\mathcal{S}_{2n}^{r,\geq}$ establishes a one-to-one correspondence with ordered trees in $\mathcal{T}_{2n+2}^{\star}$ having all its branches of even length.*

Corollary 2.5. *The restriction of ϕ to $\mathcal{R}_n^{r,\geq}$ establishes a one-to-one correspondence with ordered trees in $\mathcal{T}_{n+2}^{\star}$ where the rightmost branch starting from the root is of length 2.*

Since many bijections are already known [4, 9, 15, 22] between ordered trees and Dyck paths, it becomes easy to obtain constructive bijections between the two sets $\mathcal{D}_n$ and $\mathcal{S}_{2n}^{r,\geq}$ using Corollary 2.4.

3. Motzkin Paths Constrained by Height

In this section, we enumerate the set $\mathcal{M}_n^{h,\geq}$ of Motzkin paths of length $n \geq 0$ with a first return decomposition satisfying the two conditions (C_1) and (C_2) where $\diamond$

$M \in \mathcal{M}_n^{r,\geq}$	$\phi(M) \in \mathcal{T}_{n+2}^{\star}$
Number of returns	$\delta(\phi(M))$ (see Theorem 2.2)
Number of up steps	Number of branches minus one
Number of flats steps	$n+2-2\times$ Number of branches
Number of flats on the x-axis	Length minus 2 of the rightmost branch starting from root
Maximal length of a run of flats	Maximal length of a branch minus 2
Number of maximal runs of flats of even length	Number of branches of even length ≥ 4
Number of maximal runs of flats of odd length	Number of branches of odd length ≥ 3
Number of valleys DU	Number of branch nodes
Number of peaks UF^kD	Number of branch nodes plus one
Number of up steps minus number of valleys DU	Number of leaves minus one

Table 2: Statistic correspondences by the bijection ϕ

is the height statistic h. For $k \geq 0$, let $A_k(x) = \sum_{n\geq 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\geq 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of Motzkin paths in $\mathcal{M}_n^{h,\geq}$ having a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum_{i=0}^{k} A_i(x)$ and the generating function for the set $\mathcal{M}^{h,\geq}$, namely $M_h(x)$, is given by $M_h(x) = \lim_{k\to\infty} B_k(x)$.

Any non empty Motzkin path M of height $k \geq 0$ in $\mathcal{M}_n^{h,\geq}$, $n \geq 1$, is either F^n, or $U\alpha D\beta$, where α (resp. β) is a Motzkin path in $\mathcal{M}^{h,\geq}$ of height $k-1$ (resp. of height at most k). So we have

$$\begin{cases} A_0(x) & = B_0(x) = \frac{1}{1-x} \\ A_k(x) & = x^2 A_{k-1}(x) \cdot B_k(x) \end{cases}.$$

On the other hand, any Motzkin path of height k in $\mathcal{M}^{h,\geq}$ can be constructed from a Dyck path of height k in $\mathcal{D}^{h,\geq}$ by possibly adding flat steps before down steps, or at the end. Using the work in [3] on $\mathcal{D}^{h,\geq}$, we easily deduce $A_k(x) = \frac{1}{1-x}C_k(x^2/(1-x))$ and $B_k(x) = \frac{1}{1-x}D_k(x^2/(1-x))$, where $C_k(x)$ (resp. $D_k(x)$) is the generating function for Dyck paths of height k (resp. at most k) in $\mathcal{D}^{h,\geq}$. Therefore, we directly obtain Theorem 3.1 and Lemma 3.2 (the proofs are obtained *mutatis mutandis* as in [3]).

Theorem 3.1. *We have* $A_0(x) = B_0(x) = \frac{1}{1-x}$, $B_1(x) = \frac{1}{1-x-x^2}$, *and*

$$B_k(x) = \frac{1}{1-x} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{-1} \text{ for } k \geq 1,$$

$$M_h(x) = \frac{1}{1-x} \cdot \prod_{i=0}^{\infty} (1 - x^2 A_i(x))^{-1},$$

$$A_k(x) = \frac{x^{2k}}{(1-x)^{k+1}} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{i-k}.$$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1		1	2	4	7	12	20	33	54	88
2				1	3	8	18	39	81	165
3						1	4	13	35	88
4								1	5	19
5										1
$\sum$	1	2	3	6	11	22	43	87	176	362

Table 3: Number $a_{n,k}$ of Motzkin paths of height k in $\mathcal{M}_n^{h,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 5$.

Lemma 3.2. *For* $k \geq 1$, *we have*

$$B_k(x) = \frac{1 + x^2 B_{k-1}(x)}{1 - x - x^4 B_{k-1}(x)}.$$

Note that by taking the limits when k converges to infinity, one gets $x^4 M_h(x)^2 + (x^2 + x - 1) M_h(x) + 1 = 0$, and we retreive that $M_h(x) = \frac{1}{1-x} \cdot M\left(\frac{x^2}{1-x}\right)$. Now we show how $B_k(x), k \geq 0$, can be expressed as a closed form. Let us define the function

$$f : u \mapsto \frac{1}{1 - x - x^2 - x^4 u}.$$

For $n \geq 1$, we denote by f^n the function recursively defined by $f^n(u) = f\big(f^{n-1}(u)\big)$ anchored with $f^0(u) = u$. Lemma 3.2 induces that for any $k \geq 2$, $B_k(x) = f(B_{k-2}(x))$, which induces Theorem 3.3.

Theorem 3.3. *For $k \geq 1$, we have*

$$B_k(x) = f^{\lfloor \frac{k}{2} \rfloor}\left(B_{k \bmod 2}(x)\right)$$

with the initial cases $B_0(x) = \frac{1}{1-x}$ and $B_1(x) = \frac{1}{1-x-x^2}$.

Table 3 presents the first values of $a_{n,k}$. Whenever $k = 1$, we have $B_1(x) = 1/(1 - x - x^2)$ which is the generating function of the sequence of Fibonacci (see A000045 in [20]). Note that, by taking limits in Theorem 3.3, we retrieve the continued fraction given by P. Barry for the sequence A026418 in [20]:

$$M_h(x) = \cfrac{1}{1 - x - x^2 - \cfrac{x^4}{1 - x - x^2 - \cfrac{x^4}{1 - x - x^2 - \cfrac{x^4}{\cdots}}}}.$$

Now, we conclude this section by giving a constructive bijection between $\mathcal{M}_n^{h,\geq}$ and the set $\mathcal{T}_{n+2}^{\star}$, $n \geq 0$. Let M be a Motzkin path in $\mathcal{M}_n^{h,\geq}$ and $\alpha, \beta \in \mathcal{M}^{h,\geq}$, we recursively define the map ψ as follows:

(i) if $M = \epsilon$, then $\psi(M) = ee$,

(ii) if $M = \alpha F$ with $\psi(\alpha) = (L, R)$, then $\psi(M) = (L, eR)$,

(iii) if $M = \alpha U F^k D$ with $k \geq 0$ and $\psi(\alpha) = (L, R)$, then $\psi(M) = ((L, e^k R), ee)$,

(iv) if $M = \alpha U U \beta D \gamma D$ with $\psi(\alpha\gamma) = (L, R)$, then $\psi(M) = (L, ee(R, \psi(\beta)))$.

An illustration of this map is given in Figure 4. Notice that the two first cases (i) and (ii) are identical to those of ϕ. See Figure 5 for an example.

Theorem 3.4. *The map ψ is a bijection from $\mathcal{M}_n^{h,\geq}$ to $\mathcal{T}_{n+2}^{\star}$.*

Proof. We proceed by induction on n. Obviously, for $n = 1$, we have $\psi(F) = eee$. For $k \leq n$, we assume that ψ is a bijection from $\mathcal{M}_k^{h,\geq}$ to $\mathcal{T}_{k+2}^{\star}$ and we prove the result for $n+1$. Since $\mathcal{M}_{n+1}^{h,\geq}$ and $\mathcal{T}_{n+3}^{\star}$ have the same cardinality, it suffices to prove that if $M, M' \in \mathcal{M}_{n+1}^{h,\geq}$ with $\psi(M) = \psi(M')$ then $M = M'$. By a simple observation of Figure 4 and in a same way as for the proof of Theorem 2.2, the condition $\psi(M) = \psi(M')$ implies that M and M' belong to the same case (i), (ii), (iii), or (iv). For the first two cases, it is straightforward that the recurrence hypothesis induces $M = M'$. For the case (iii), we have $\psi(M) = \psi(M') = ((L, e^k R), ee)$. The first branch of R is necessarily of length two, otherwise this would mean that the first branch of R is of length at least three, and thus α would have a flat step at its end, which is impossible for $M \in \mathcal{M}^{h,\geq}$ satisfying (iii). Thus, k is entirely

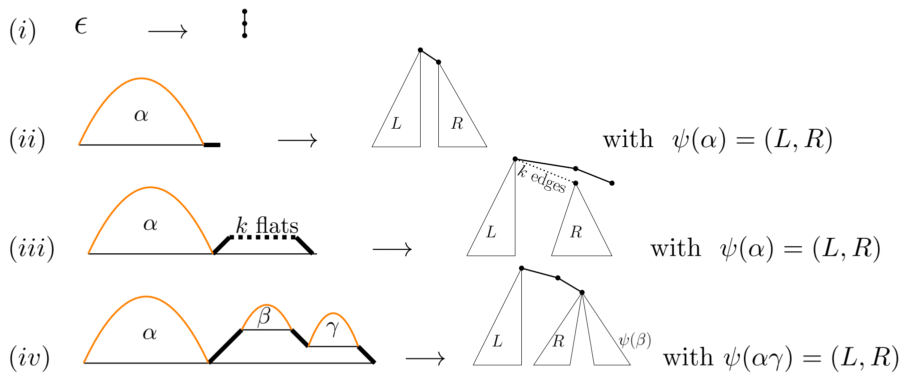

Figure 4: An illustration of the bijection ψ.

determined by $\psi(M) = \psi(M')$, and the recurrence hypothesis induces $M = M'$. For the case (iv), we have $M = \alpha UU\beta D\gamma D$, $M' = \alpha' UU\beta' D\gamma' D$ and $\psi(M) = \psi(M')$. This implies that $\psi(\beta) = \psi(\beta')$, $\psi(\alpha\gamma) = \psi(\alpha'\gamma')$, and the recurrence hypothesis gives $\beta = \beta'$ and $\alpha\gamma = \alpha'\gamma'$. Since $h(\alpha) \geq h(\beta) + 2 \geq h(\gamma) + 1$, $\alpha\gamma$ has a unique decomposition satisfying these inequalities. So we conclude $\alpha = \alpha'$ and $\gamma = \gamma'$. $\square$

The bijection ψ establishes correspondences between several statistics on Motzkin paths and ordered trees. Table 4 summarizes the main correspondences. We do not succeed to determine the statistic on $\mathcal{T}_n^\star$ which is the image by ψ of h. Note that using a straightforward proof by induction, the number of branch nodes of $\psi(M)$ equals the sum of $\lfloor \frac{k}{2} \rfloor$ on all maximal runs U^k in M. As a byproduct and using Corollary 3.5, Tables 2 and 4 induce several bijective correspondences between statistics on $\mathcal{M}_n^{r,\geq}$ and $\mathcal{M}_n^{h,\geq}$. See Figure 5 for an example of such a correspondence.

Corollary 3.5. *The bijection $\psi^{-1} \circ \phi$ induces one-to-one correspondences between $\mathcal{M}_n^{r,\geq}$ and $\mathcal{M}_n^{h,\geq}$, $\mathcal{D}_n^{r,\geq}$ and $\mathcal{D}_n^{h,\geq}$, $\mathcal{S}_n^{r,\geq}$ and $\mathcal{S}_n^{h,\geq}$, $\mathcal{R}_n^{r,\geq}$ and $\mathcal{R}_n^{h,\geq}$. These bijections preserve the number of up steps, the number of flats, the number of flats on the x-axis, the number of maximal runs of flats of even length, the maximal length of a run of flats, and it transports the number of valleys DU into the sum of $\lfloor \frac{k}{2} \rfloor$ on all maximal runs U^k.*

4. Going Further With 2-Motzkin Paths

In this part, we extend the previous study to 2-Motzkin paths. A 2-Motzkin path is a Motzkin path where flat steps can be of two kinds: F for straight and K for wavy. The number of 2-Motzkin paths with n steps is given by the $(n+1)$-th Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$. We refer to [8, 12] for a constructive bijection between

$M \in \mathcal{M}_n^{h,\geq}$	$T = \psi(M) \in \mathcal{T}_{n+2}^{\star}$
Height	Open question
Number of up steps	Numbers of branches minus one
Number of flats	$n+2-2\times$ Number of branches
Number of flats on the x-axis	Length minus 2 of the rightmost branch starting from root
Number of maximal runs of flats of even length	Number of branches of even length ≥ 4
Number of maximal runs of flats of odd length	Number of branches of odd length ≥ 3
Maximal length of a run of flats	Maximal length of a branch minus 2
$\sum\limits_{\text{Maximal } U^k} \lfloor \frac{k}{2} \rfloor$	Number of branch nodes
Number of up steps minus $\sum\limits_{\text{Maximal } U^k} \lfloor \frac{k}{2} \rfloor$	Number of leaves minus one

Table 4: Statistic correspondences by the bijection ψ

2-Motzkin paths of length n and Dyck paths of semilength $n+1$. For a statistic $s \in \{r, h\}$, we define the set $\bar{\mathcal{M}}_n^{s,\geq}$ of 2-Motzkin paths of length n satisfying the two conditions (C_1) and (C_2) given in Introduction and we set $\bar{\mathcal{M}}^{s,\geq} = \bigcup_{n\geq 0} \bar{\mathcal{M}}_n^{s,\geq}$. As we did for Motzkin paths, a 2-Motzkin path in $\bar{\mathcal{M}}^{s,\geq}$ can be constructed from a Dyck path in $\mathcal{D}^{s,\geq}$ by possibly adding flat steps (F or K) before any down step, or at the end. So, the generating function $\bar{M}_s(x)$ for the set $\bar{\mathcal{M}}^{s,\geq}$ is:

$$\bar{M}_s(x) = \frac{1}{1-2x} \cdot M\left(\frac{x^2}{1-2x}\right).$$

The first coefficients of its Taylor expansion are $1, 2, 5, 12, 30, 76, 196, 512, 1353, 3610, 9713, 26324$, and these numbers correspond to the first differences of Motzkin numbers, which are also called generalized Ballot numbers (see `A002026` in [20]).

4.1. Enumerative Results

Below, we state enumerative results for $s \in \{r, h\}$. We do not give the proofs since it suffices to reread Sections 2 and 3 by replacing all fractions $\frac{1}{1-x}$ with $\frac{1}{1-2x}$ in all functional equations in order to taking into account the two kinds of flats. Theorem 4.1 and Table 5 deal with the statistic of the number of returns, while Theorem 4.2 and Table 6 treat the case of the height.

Let $A(x,y) = \sum\limits_{k,n\geq 0} a_{n,k} x^n y^k$ be the generating function where the coefficient

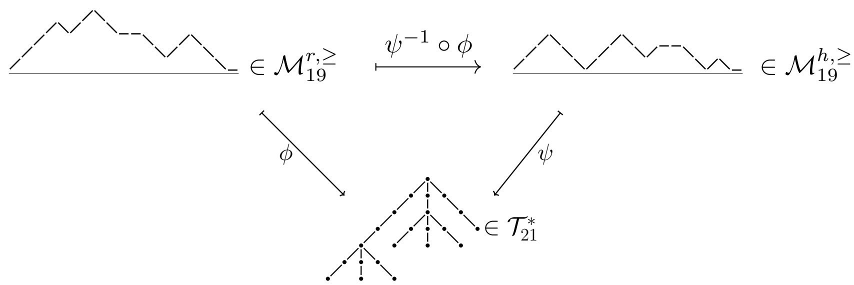

Figure 5: Example of one-to-one correspondence between two Motzkin paths in $\mathcal{M}_{19}^{r,\geq}$ and $\mathcal{M}_{19}^{h,\geq}$ passing by a tree in $\mathcal{T}_{21}^{\star}$.

$a_{n,k}$ of $x^n y^k$ is the number of 2-Motzkin paths with k returns in $\bar{\mathcal{M}}_n^{r,\geq}$.

Theorem 4.1. *The generating function $A(x, y)$ is given by*

$$A(x, y) = A_0(x) + A_1(x)y + A_2(x)y^2$$

with

$$A_0(x) = \frac{1}{1-2x}, \qquad A_1(x) = \frac{1-2x-x^2-\sqrt{1-4x+2x^2+4x^3-3x^4}}{2x^2\,(1-2\,x)}, \text{ and}$$

$$A_2(x) = \frac{\left(1-2x-x^2-\sqrt{1-4x+2x^2+4x^3-3x^4}\right)^2}{4x^4\,(1-2\,x)}.$$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
0	2	4	8	16	32	64	128	256	512	1024
1		1	4	13	38	106	288	772	2056	5465
2				1	6	26	96	325	1042	3224
$\sum$	2	5	12	30	76	196	512	1353	3610	9713

Table 5: Number $a_{n,k}$ of 2-Motzkin paths with k returns in $\bar{\mathcal{M}}_n^{r,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 2$. For $k \geq 3$ and $n \geq 1$, we have $a_{n,k} = 0$.

For $k \geq 0$, let $A_k(x) = \sum_{n\geq 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\geq 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of 2-Motzkin paths of height k (resp. at most k) in $\bar{\mathcal{M}}_n^{h,\geq}$.

Theorem 4.2. *We have* $A_0(x) = B_0(x) = \frac{1}{1-2x}$, $B_1(x) = \frac{1}{1-2x-x^2}$, *and*

$$B_k(x) = \frac{1}{1-2x} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{-1} \text{ for } k \geq 1,$$

$$\bar{M}_h(x) = \frac{1}{1-2x} \cdot \prod_{i=0}^{\infty} (1 - x^2 A_i(x))^{-1},$$

$$A_k(x) = \frac{x^{2k}}{(1-2x)^{k+1}} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{i-k}.$$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	2	4	8	16	32	64	128	256	512	1024
1		1	4	13	38	105	280	729	1866	4717
2				1	6	26	96	324	1032	3159
3						1	8	43	190	748
4								1	10	64
5										1
$\sum$	2	5	12	30	76	196	512	1353	3610	9713

Table 6: Number $a_{n,k}$ of Dyck paths of height k in $\bar{\mathcal{M}}_n^{h,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 5$.

Note that $B_1(x)$ is the generating function of the sequence of Pell numbers (`A000129` in [20]) which count the left factors of Grand Schröder paths.

Lemma 4.3. *For* $k \geq 1$, *we have*

$$B_k(x) = \frac{1 + x^2 B_{k-1}(x)}{1 - 2x - x^4 B_{k-1}(x)}.$$

Note that by taking the limits when k converges to infinity, one gets $x^4 \bar{M}_h(x)^2 + (x^2 + 2x - 1)\bar{M}_h(x) + 1 = 0$, and we retrieve that $\bar{M}_h(x) = \frac{1}{1-2x} \cdot M(\frac{x^2}{1-2x})$. Now, we show how $B_k(x)$ can be expressed as a closed form. Let us define the function

$$g : u \mapsto \frac{1}{1 - 2x - x^2 - x^4 u}.$$

Lemma 4.3 induces that for any $k \geq 2$, $B_k(x) = g(B_{k-2}(x))$, which induces Theorem 4.4.

Theorem 4.4. *For $k \geq 1$, we have*

$$B_k(x) = g^{\lfloor \frac{k}{2} \rfloor}\left(B_{k \bmod 2}(x)\right)$$

with the initial cases $B_0(x) = \frac{1}{1-x}$ and $B_1(x) = \frac{1}{1-2x-x^2}$.

By taking limits in Theorem 4.4, we obtain the continued fraction for the sequence A002026 in [20]:

$$\bar{M}_h(x) = \cfrac{1}{1 - 2x - x^2 - \cfrac{x^4}{1 - 2x - x^2 - \cfrac{x^4}{1 - 2x - x^2 - \cfrac{x^4}{\cdots}}}}.$$

4.2. Constructive Bijection

In this part, we construct a bijection between $\bar{\mathcal{M}}_n^{s,\geq}$, $s \in \{r, h\}$, and the subset $\mathcal{B}_{n+2}$ of $\mathcal{T}_{n+2}$ consisting of ordered trees with $n+2$ edges where the root has two children, and where every node has at most two children. In order to do this, we firstly extend the previous bijections ϕ and ψ respectively on the sets $\bar{\mathcal{M}}_n^{r,\geq}$ and $\bar{\mathcal{M}}_n^{h,\geq}$ by setting $\phi(\alpha K) = \psi(\alpha K) = (L, \bar{e}R)$ whenever $\phi(\alpha) = \psi(\alpha) = (L, R)$, and where e is a straight edge and $\bar{e}$ is a wavy edge. So, these two maps establish one-to-one correspondences between $\bar{\mathcal{M}}_n^{s,\geq}$, $s \in \{r, h\}$, and the set $\mathcal{T}_{n+2}^{\star\star}$ of ordered trees with $n+2$ edges having no branches of length one, and such that any branch is constituted of edges of two kinds (straight and wavy) except for its two last edges which must be straight. Secondly, we recursively define a map χ between $\mathcal{T}_{n+2}^{\star\star}$ and the subset $\mathcal{B}_{n+2}$.

For $n \geq 2$, let T be an ordered tree in $\mathcal{T}_n^{\star\star}$ and $\alpha, \beta \in \mathcal{T}^{\star\star} = \bigcup_{n\geq 2} \mathcal{T}_n^{\star\star}$ (which induces that α and β are non-empty). Let $\mathcal{T}^{\star\star(1)}$ be the set of ordered trees in $\mathcal{T}^{\star\star}$ where the root has only one child, and let $\mathcal{T}^{\star\star(2)} = \mathcal{T}^{\star\star} \backslash \mathcal{T}^{\star\star(1)}$.

We define χ as follows:
In the case where $T \in \mathcal{T}^{\star\star(1)}$:

(i) if $T = ee$, then $\chi(T) = (e, e)$,

(ii) if $T = ee\alpha$ with $\alpha \in \mathcal{T}^{\star\star(2)}$, then $\chi(T) = (e\chi(\alpha), e)$,

(iii) if $T = e\alpha$ with $\alpha \in \mathcal{T}^{\star\star(1)}$, then $\chi(T) = (L, e^{k+1})$ for $\chi(\alpha) = (L, e^k)$, $k \geq 1$,

(iv) if $T = \bar{e}\alpha$ with $\alpha \in \mathcal{T}^{\star\star(1)}$, then $\chi(T) = (e(L, e^{k-1}), e)$ for $\chi(\alpha) = (L, e^k)$, $k \geq 1$.

In the case where $T \in \mathcal{T}^{\star\star(2)}$:

(*v*) if $T = (\alpha, \beta)$ with $\alpha \in \mathcal{T}^{\star\star}$, $\beta \in \mathcal{T}^{\star\star(1)}$, then $\chi(T) = (L, e^k\chi(\alpha))$ for $\chi(\beta) = (L, e^k), k \geq 1$.

An illustration of this map is given in Figure 6.

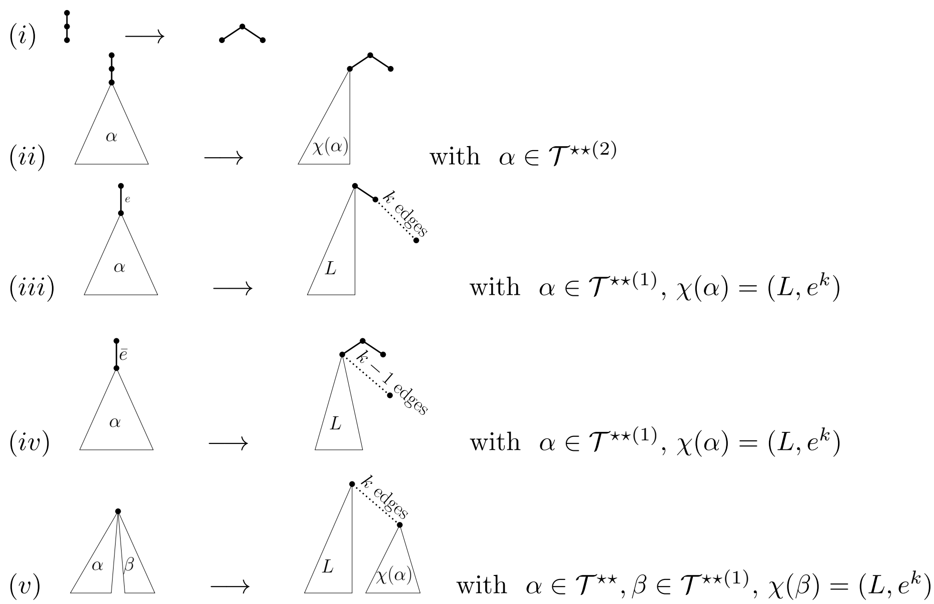

Figure 6: An illustration of the bijection χ.

It is easy to observe the following facts.

Fact 1. *The image by* χ *of* $T \in \mathcal{T}^{\star\star(1)}$ *does not have any branch node in its right part, that is, its right branch is of the form* e^k *for* $k \geq 1$.

Fact 2. *The image of* $T \in \mathcal{T}^{\star\star(2)}$ *has at least one branch node in its right part.*

Fact 3. *The image of a tree satisfying (iii) has a branch of length at least two as right part of the root, while the image of a tree satisfying (i), (ii) or (iv) has a branch of length one.*

Fact 4. *The image of a tree satisfying (ii) has a left subtree where the root has two subtrees such that the rightmost contains at least one branch node.*

Fact 5. *The image of a tree satisfying (iv) has a left subtree where either the root has two subtrees such that the rightmost is a branch (whenever* $k \geq 2$*), or the root has only one subtree (whenever* $k = 1$*).*

Theorem 4.5. *For* $n \geq 2$*, the map* χ *is a bijection from* $\mathcal{T}_n^{\star\star}$ *to* $\mathcal{B}_n$.

Proof. Considering Facts 1-5, the equality $\chi(T) = \chi(T')$ implies that T and T' belong to the same case (i), (ii), (iii), (iv) or (v). The injectivity is obtained by a simple induction. Now, it suffices to check that $\mathcal{B}_n$ and $\mathcal{T}_n^{\star\star}$ have the same cardinality. Let $E(x)$ be the generating function for $\mathcal{B}_n$, $n \geq 2$, and $E_1(x)$ the generating function for the set $\mathcal{B}^1 = \bigcup_{n\geq 2} \mathcal{B}_n^1$ of trees in $\mathcal{B}$ having a branch as right part. We set $\mathcal{B}^2 = \mathcal{B}\backslash\mathcal{B}^1$ and $E_2(x) = E(x) - E_1(x)$. A tree T in $\mathcal{B}^1$ consists of a branch of length at least one for its right part, and a unary-binary tree connected to the root for its left subtree. So, we have $E_1(x) = \frac{x^2}{1-x}M(x)$, where $M(x)$ is the Motzkin generating function for the unary-binary ordered trees. A tree T in $\mathcal{B}^2$ consists of a right part which is a branch of length at least one connected to a tree in $\mathcal{B}$, and a unary-binary tree connected to the root as left subtree. So, we deduce $E_2(x) = \frac{x^2}{1-x}E(x)M(x)$. A simple calculation of $E(x)$ proves that $\mathcal{B}_n$ and $\mathcal{T}_n^{\star\star}$ have the same cardinality for $n \geq 2$, which completes the proof. □

Remark 4.6. The set $\mathcal{B}$ is clearly in bijection with the square of the set of unary-binary trees (*i.e.*, ordered trees where any node, including the root, has at most two children). Then the generating function for $\mathcal{B}$ is given by $x^2M(x)^2$, where $M(x)$ is the generating function for the Motzkin numbers. So, the bijection $\psi \circ \chi$ establishes a bijective proof of the (apparently new) equality

$$M(x)^2 = \frac{1}{1-2x} \cdot M\left(\frac{x^2}{1-2x}\right).$$

Corollary 4.7. *The restriction of χ to $\mathcal{T}_n^\star$ establishes a one-to-one correspondence with ordered trees in $\mathcal{B}_n$ such that any left node has zero or two children, and any left leaf has a right sibling having at most one child.*

Proof. The restriction of χ to $\mathcal{T}_n^\star$ is defined using cases (i), (ii), (iii) and (v). Let $T \in \mathcal{T}_n^\star$ and $T' = \chi(T)$. By a simple induction, we observe that any left node of T' has zero or two children, and any left leaf has a right sibling having at most one child. Now, it suffices to prove that these trees are counted by the sequence `A026418` in [20] (as for $\mathcal{T}_n^\star$). Let $E(x)$ be the generating function for these trees. Since such a tree is of one of the forms (e, e), (eS, e), (eS, eS') and (L, eR) whenever $S = (L, R)$ and S' are also such trees, then $E(x)$ satisfies the functional equation: $E(x) = x^2 + x^2E(x) + xE(x) + x^2E(x)^2$. A simple calculation provides $E(x) = x^2M(x^2/(1-x))/(1-x)$ which achieves the proof. □

Corollary 4.8. *The restriction of χ to the subset of $\mathcal{T}_{2n}^\star$ consisting of ordered trees having n branches establishes a one-to-one correspondence with binary ordered trees in $\mathcal{B}_{2n}$ such that any node has zero or two children, and any left leaf has a right leaf as sibling.*

Proof. The proof is obtained as for the previous corollary without considering the case (iii). □

5. Conclusion and Future Works

In Corollary 3.5, we obtain a constructive bijection between $\mathcal{D}_n^{r,\geq}$ and $\mathcal{D}_n^{h,\geq}$. Unfortunately, this bijection does not transport simply the two statistics r and h. Is there another more natural bijection that behaves well with these statistics? On the other hand, we have seen that Schröder paths in $\mathcal{S}_{2n}^{r,\geq}$ and $\mathcal{S}_{2n}^{h,\geq}$ are enumerated by the n-th Catalan numbers. Is it possible to obtain direct bijections between these sets and the set of Dyck paths of semilength n, by not passing by a set of ordered trees? More generally, can we obtain bijective correspondences between these sets and other combinatorial classes such as permutations?

It would be interesting to study several parameters or statistics on the sets $\mathcal{M}_n^{s,\geq}$ and $\mathcal{D}_n^{s,\geq}$ for $s \in \{r, h\}$. For instance, the distribution of a given pattern is a subject widely studied these last years (in particular the avoidance of a pattern). Also, we could enumerate the restricted sets of paths having a non-decreasing height sequence (see [1, 6, 7]).

Can one develop algorithm for uniform random generation of an element of $\mathcal{M}_n^{s,\geq}$? Can one develop an efficient algorithm for the exhaustive generation of these objects?

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