Gray codes for Fibonacci q-decreasing words

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Abstract. A length n binary word is q-decreasing, $q \geqslant 1$, if every of its maximal factors of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$. In particular, in 1-decreasing words every run of 0s is immediately followed by a strictly shorter run of 1s. We show constructively that these words are in bijection with binary words having no occurrences of 1^{q+1}, and thus they are enumerated by the $(q + 1)$-generalized Fibonacci numbers. We give some enumerative results and reveal similarities between q-decreasing words and binary words having no occurrences of 1^{q+1} in terms of the frequency of 1-bits. In the second part of our paper, we provide an efficient exhaustive generating algorithm for q-decreasing words in lexicographic order, for any $q \geqslant 1$, show the existence of 3-Gray codes and explain how a generating algorithm for these Gray codes can be obtained. Moreover, we give the construction of a more restrictive 1-Gray code for 1-decreasing words, which in particular settles a conjecture stated recently in the context of interconnection networks by Eğecioğlu and Iršič.

1   Introduction and preliminaries

The Fibonacci sequence origins have been traced back to the works of ancient Indian mathematician Ācārya Piṅgala dealing with rhythmic structure patterns in Sanskrit poetry [25], [16, p. 50]. Over time, the study of words and patterns became more abstract and systematic (see for instance Lothaire's books [17,18,19] and [7]). A considerable amount of questions concerning efficient enumeration and generation of words respecting certain properties (including pattern avoidance) were mathematically formulated and answered only relatively recently, the works closely related to the present study include [3,5,9,10,12,27,28,29].

In this paper we introduce q-decreasing words, a novel class of run-restricted binary words enumerated by the $(q + 1)$-generalized Fibonacci numbers, $q \geqslant 1$. For $q = 1$ the subclass of such words that start with 0 was recently considered in the context of induced subgraphs of hypercubes [9,10]. The q-decreasing words could be interesting objects to study in other domains, for example in stringology, the study of string algorithms [8]. In Section 2 we present a bijection between this novel class of words and Fibonacci words, i.e., binary words avoiding consecutive 1s. Section 3 is devoted to the presentation of several generating functions and enumeration results. Finally, in Section 4, we show the existence of a 3-Gray code for any $q \geqslant 1$, give an efficient exhaustive generating algorithms

and a much more involved construction for a 1-Gray code in the special case $q = 1$. In particular, the latter Gray code gives a Hamiltonian path in Fibonacci-run graphs whose existence is conjectured in [9].

The following set of notations is adopted. Let $\mathcal{B}$ denote the set of all finite-length binary words, i.e., strings over the alphabet $\{0, 1\}$, and $\mathcal{B}_n$, $n \geqslant 0$, be the set of all binary words of length n. For a given binary word w we use the notation w_i to mean the letter at position i.

A nonempty sequence of adjacent letters inside a word is called *factor*. A factor v repeated k times is denoted by v^k, for instance $(00)^2 1^2 = 000011$. For a given length n, the notation v^* is used to repeat factor v as many times as possible, until the length n is reached, possibly trimming extra symbols at the end; and the length n will be understood from the context. For example, if a word v of length $n = 7$ is equal to $(001)^*$ it means $v = 0010010$.

The set of all n-length binary words containing no occurrences of factor v is denoted by $\mathcal{B}_n(v)$. Let $\mathcal{B}(v) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(v)$. The concatenation of two words w and v is denoted by $w \cdot v$ or simply by wv. If v is a binary word and $\mathcal{W}$ is a set of binary words, let $\mathcal{W} \cdot v = \{w \cdot v \mid w \in \mathcal{W}\}$, and $v \cdot \mathcal{W}$ is defined similarly. Whenever $\mathcal{A}$ and $\mathcal{C}$ are two subsets of $\mathcal{B}$, we define $\mathcal{A} \cdot \mathcal{C} = \{a \cdot c \mid a \in \mathcal{A}, c \in \mathcal{C}\}$.

Following [20] the nth k-generalized Fibonacci number is defined as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant k - 2, \\ 1 & \text{if } n = k - 1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases} \tag{1}$$

As noted in [28], the generating function $g_k(x) = \sum_{n=0}^{\infty} f_{n,k} x^n$ for k-generalized Fibonacci numbers is

$$g_k(x) = \frac{x^{k-1}}{1 - x - x^2 - \cdots - x^k} = \frac{x^{k-1} - x^k}{1 - 2x + x^{k+1}}. \tag{2}$$

Related constructions also appear in [11, p. 42] and in [1, p. 309].
Classical fact. *The number of words in $\mathcal{B}_n(1^k)$ equals $f_{n+k,k}$ for $k \geqslant 2$, moreover*

$$\mathcal{B}_n(1^k) = \begin{cases} \mathcal{B}_n & \text{if } n < k, \\ \bigcup_{i=0}^{k-1} 1^i 0 \cdot \mathcal{B}_{n-i-1}(1^k) & \text{otherwise.} \end{cases} \tag{3}$$

The classical fact comes, for instance, from [15, p. 286]. The binary words avoiding two consecutive 1s are counted by Fibonacci numbers, words without factor 111 are counted by Tribonacci numbers, etc. We call such words *(generalized) Fibonacci words*. The On-line Encyclopedia of Integer Sequences founded by N.J.A. Sloane [26] contains several corresponding sequences (see for example A000045 and A000073, after taking a binary complement). Gray codes for Fibonacci words are discussed in [28], and more generally, Gray codes for words avoiding a given factor in [3,6,24,27].

Lemma 1. *For $q \geqslant 1$, the bivariate generating function $B_q(x,y) = \sum_{n,k\geqslant 0} b_{n,k} x^n y^k$ where the coefficient $b_{n,k}$ of $x^n y^k$ is the number of Fibonacci words of length n having k 1s in $\mathcal{B}_n(1^{q+1})$ is*

$$B_q(x,y) = \frac{y\left(1 - (xy)^{q+1}\right)}{y - xy^2 - xy + (xy)^{q+2}}.$$

Proof. Alternatively to (3), the set $\mathcal{B}(1^{q+1})$ of (any length) binary words avoiding 1^{q+1} can be defined recursively as

$$\mathcal{B}(1^{q+1}) = \mathbb{1}_q \cup \bigcup_{i=0}^{q} 1^i 0 \cdot \mathcal{B}(1^{q+1})$$

where $\mathbb{1}_q = \bigcup_{i=0}^{q} \{1^i\}$. It follows that the bivariate generating function $B_q(x,y)$ satisfies the functional equation

$$B_q(x,y) = \sum_{i=0}^{q} x^i y^i + B_q(x,y) \sum_{i=0}^{q} x^{i+1} y^i,$$

and we have $B_q(x,y) = \frac{y\left(1-(xy)^{q+1}\right)}{y-xy^2-xy+(xy)^{q+2}}$. $\qquad\square$

It is not surprising that $B_q(x,1) = \frac{g_{q+1}(x)-x^q}{x^{q+1}}$, see the (2). Indeed, both sides of this equality are generating functions for the sequence $(f_{n+q+1,q+1})_{n\geqslant 0}$ counting words in $\mathcal{B}(1^{q+1})$, which is the $(q+1)$th left shift of the sequence $(f_{n,q+1})_{n\geqslant 0}$, see (1) and the classical fact following it.

The *Hamming distance* between two binary words of the same length equals the number of positions at which they differ. A k-Gray code for a set $\mathcal{A} \subset \mathcal{B}_n$ is an ordered list $\mathbf{A}$ for $\mathcal{A}$, such that the Hamming distance between any two consecutive words in $\mathbf{A}$ is at most k, and we say that words in $\mathbf{A}$ are listed in *Gray code order*. Frank Gray's patent [12] discusses an early example and application of such a code for the set of n-length binary words. The concatenation of two ordered lists $\mathbf{L}_1$ and $\mathbf{L}_2$ is denoted by $\mathbf{L}_1 \circ \mathbf{L}_2$, and $\overleftarrow{\mathbf{L}}$ designates the reverse of the list $\mathbf{L}$. If $\mathbf{L}$ is a list of words, then $\langle \mathbf{L} \rangle^i = \mathbf{L}$ whenever i is even, and $\langle \mathbf{L} \rangle^i = \overleftarrow{\mathbf{L}}$ otherwise. First and last elements of $\mathbf{L}$ are denoted respectively by $\mathrm{first}(\mathbf{L})$ and $\mathrm{last}(\mathbf{L})$. Also, we denote by $\widehat{\mathbf{L}}$ the list obtained from $\mathbf{L}$ by deleting its first element.

For example, a list containing elements 00, 01 and 11 will be noted as $\mathbf{L} = 00 \circ 01 \circ 11$, and $\widehat{\mathbf{L}} = 00 \circ 01$.

Definition 1. *A binary word is called q-decreasing, for $q \geqslant 1$, if any of its length maximal factors of the form $0^a 1^b$, $a > 0$, satisfies $q \cdot a > b$.*

The set of q-decreasing words of length n is denoted by $\mathcal{W}_n^q$. For example we have $\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$. See also Table 1 for the sets $\mathcal{W}_4^2$ and $\mathcal{W}_6^1$. Let $\mathcal{W}^q = \bigcup_{n=0}^{\infty} \mathcal{W}_n^q$.

2　Bijection with classical Fibonacci words

In this section we prove that q-decreasing words, $q \geqslant 1$, are enumerated by $(q + 1)$-generalized Fibonacci numbers defined in (1). We start with a definition and several propositions.

Definition 2. *For any $q \geqslant 1$, we define the map ψ^q from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$ as*

$$\psi^q(w) = \begin{cases} v001^{k+q} & \text{if } w = v01^k, v \in \mathcal{B}, k \geqslant 0, \\ 1^{n+q+1} & \text{otherwise.} \end{cases}$$

Less formally, ψ^q inserts a factor 01^q immediately after the last occurrence of 0, and it adds the suffix 1^{q+1} to the word containing no 0. For example $\psi^1(0) = \mathbf{001}$, $\psi^1(00011) = 000\mathbf{0}11\mathbf{1}$, $\psi^2(0011101) = 0011100\mathbf{0}11\mathbf{1}$ and $\psi^5(1) = 1\mathbf{111111}$. The value of q will be clear from the context, so by slight abuse of notation ψ^q will be denoted ψ throughout the paper.

Proposition 1. *For $n \geqslant 0, q \geqslant 1$, ψ is an injective map from from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$.*

Proof. For two n-length words $w \neq w'$ we show that $\psi(w) \neq \psi(w')$. It is clear that if one of the given words contains no 0 the injectivity holds. Otherwise we have two cases. If $w = v01^k$ and $w' = v'01^k$ then we have necessarily $v \neq v'$ and $v001^{k+q} \neq v'001^{k+q}$, so the images are different. If $w = v01^k$ and $w' = v'01^\ell$ with $k \neq \ell$, then $v001^{k+q} \neq v'001^{\ell+q}$ and again $\psi(w) \neq \psi(w')$. $\qquad\square$

In the following, we will use the restriction of ψ to the set of q-decreasing words, namely $\psi : \mathcal{W}_n^q \to \mathcal{W}_{n+q+1}^q$. It is possible due to Proposition 2 below.

Proposition 2. *For $n \geqslant 0, q \geqslant 1$, $\psi(\mathcal{W}_n^q)$ consists of all q-decreasing words of length $n + q + 1$ ending with at least q ones.*

Proof. If $w = 1^n$, then $\psi(w) = 1^{n+q+1}$. Otherwise, we write $w = v0^a1^b$ where $a > b/q \geqslant 0$ and the word v is either empty or ends with 1. So $\psi(v0^a1^b) = v0^{a+1}1^{q+b}$. As we have $1 + a > 1 + b/q = (q + b)/q$, $\psi(w)$ is a q-decreasing word ending with at least q 1s. Similarly, any $(n + q + 1)$-length q-decreasing word ending with at least q 1s can be obtained from a (unique) word in $\mathcal{W}_n^q$ by ψ. $\quad\square$

Now, we present a one-to-one correspondence between Fibonacci and q-decreasing words. Recall that, for $q \geqslant 1$, the set $\mathcal{B}_n(1^{q+1})$ of $(q + 1)$-generalized Fibonacci words is the set of binary words of length n with no 1^{q+1} factors, see (3) for the recursive definition of these words according to their length.

Definition 3. *We define the map $\phi : \mathcal{B}_n(1^{q+1}) \to \mathcal{B}_n$ as*

$$\phi(w) = \begin{cases} 1^k & \text{if } w = 1^k \text{ and } k \in [0, q], \\ \psi\big(\phi(v)\big) & \text{if } w = 1^q0v, \\ \phi(v)01^k & \text{if } w = 1^k0v \text{ and } k \in [0, q - 1]. \end{cases} \qquad (4)$$

See Table 1(a) for the images of the words in $\mathcal{B}_4(111)$ through ϕ.

Theorem 1. *For $n \geqslant 0, q \geqslant 1$, ϕ maps bijectively $\mathcal{B}_n(1^{q+1})$ into $\mathcal{W}_n^q$.*

Proof. We proceed by induction on n. The classical decomposition in (3) gives rise to the three cases specified in (4), and we have:
(i) If $n \leqslant q$, then the word 1^n is sent by ϕ to 1^n;
(ii) Words of the form $1^q 0v$, where $v \in \mathcal{B}_{n-q-1}(1^{q+1})$, are sent to words ending by at least q 1s (see Proposition 2);
(iii) Words of the form $1^k 0v$, where $k \in [0, q-1]$ and $v \in \mathcal{B}_{n-k-1}(1^{q+1})$, are sent to words ending by at most $q-1$ 1s.
Using the bijectivity of ψ (see Propositions 1 and 2) and the induction hypothesis, it is routine to check that $\phi(w) \in \mathcal{W}_n^q$ for any $w \in \mathcal{B}_n(1^{q+1})$, and $\phi(w) \neq \phi(w')$ for any two different words $w, w' \in \mathcal{B}_n(1^{q+1})$.

Similarly, by induction on n, any word in $\mathcal{W}_n^q$ can be obtained by ϕ from a word in $\mathcal{B}_n(1^{q+1})$, and the statement holds.   □

It follows that $(q+1)$-order Fibonacci words and q-decreasing words are equinumerous.

3   Some enumeration results

Here we provide a bivariate generating function $W_q(x, y) = \sum_{n,k \geqslant 0} w_{n,k} x^n y^k$, where $w_{n,k}$ is the number of n-length q-decreasing words having k 1s. This bivariate generating function is of a particular interest since it will help us (see Corollary 1) to prove that q-decreasing words satisfy a necessary condition for the existence of 1-Gray code, called *parity condition*. More precisely, if a set $\mathcal{A}$ of binary words admits a 1-Gray code, and $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) denotes the subset of $\mathcal{A}$ having even (resp. odd) number of 1s, then the *parity difference* $|\mathcal{A}^+| - |\mathcal{A}^-|$ must be equal to either 0, 1, or -1. Indeed, the graph where the vertex set is $\mathcal{A}$ and edges connect vertices with Hamming distance one is bipartite with partite sets $\mathcal{A}^+$ and $\mathcal{A}^-$. A Hamiltonian path in this graph (or equivalently a 1-Gray code for $\mathcal{A}$) cannot exist unless it satisfies the parity condition. This parity condition is used for instance in [27] to investigate the possibility of 1-Gray code for a set of words avoiding a given factor.

In order to derive the expression of $W_q(x, y)$, we use the following decomposition of the set $\mathcal{W}^q$:

$$\mathcal{W}^q = \mathbb{1} \cup \mathcal{W}^q \cdot \mathcal{S}^q,$$

where $\mathbb{1} = \cup_{n=0}^{\infty} \{1^n\}$ and $\mathcal{S}^q$ corresponds to the set of all factors of the form $0^a 1^b$ respecting q-decreasing property (i.e., $a > b/q \geqslant 0$) such that none of them is a concatenation of other factors respecting q-decreasing property. More precisely, a is the smallest integer strictly greater than b/q, i.e., $a = \lfloor b/q \rfloor + 1$. A factor from $\mathcal{S}^q$ will be called q-*prime factor*, and thus $\mathcal{S}^q$ is the set of such factors. For instance: $\mathcal{S}^1 = \{0, 001, 00011, 0000111, \ldots\}$, $\mathcal{S}^2 = \{0, 01, 0011, 00111, 0001111, 00011111, \ldots\}$.

Lemma 2. *For $q \geqslant 1$, the bivariate generating function $S_q(x, y) = \sum_{n,k \geqslant 0} s_{n,k} x^n y^k$ where the coefficient $s_{n,k}$ is the number of q-prime factors of length n having exactly k 1s is:*

$$S_q(x, y) = \frac{x\left(1 - (xy)^q\right)}{(xy - 1)(x^{q+1}y^q - 1)}.$$

Proof. Any q-prime factor is of the form $0^a 1^b$ with $a = \lfloor b/q \rfloor + 1$. So, if $b = kq + r$ with $k \geqslant 0$ and $r \in [0, q-1]$, then $a + b = k(q+1) + r + 1$ and we can write:

$$S_q(x, y) = \sum_{k=0}^{\infty} \sum_{r=0}^{q-1} x^{k(q+1)+r+1} y^{kq+r}.$$

A simple calculation results in the claimed formula.   □

Theorem 2. *For $q \geqslant 1$, the bivariate generating function $W_q(x, y) = \sum_{n,k \geqslant 0} w_{n,k} x^n y^k$ where the coefficient $w_{n,k}$ is the number of n-length q-decreasing words containing exactly k 1s is given by:*

$$W_q(x, y) = \frac{1 - x^{q+1} y^q}{1 - xy - x + x^{q+2} y^{q+1}}.$$

Proof. Due to the decomposition $\mathcal{W}^q = \mathbb{1} \cup \mathcal{W}^q \cdot \mathcal{S}^q$, we have $W_q(x, y) = \frac{1}{1 - xy} \cdot \frac{1}{1 - S_q(x,y)}$, and the result holds after applying Lemma 2.   □

Corollary 1. *For any $n \geqslant 0, q \geqslant 1$, the set $\mathcal{W}_n^q$ satisfies the parity condition.*

Proof. The generating function $D_q(x) = \sum_{n \geqslant 0} d_n x^n$ where the coefficient d_n is the parity difference corresponding to the set $\mathcal{W}_n^q$ is obtained by making the substitution $y = -1$ in $W_q(x, y)$:

$$D_q(x) = \frac{(-1)^q x^{q+1} - 1}{(-1)^q x^{q+2} - 1}.$$

When q is even, $D_q(x) = \frac{x^{q+1} - 1}{x^{q+2} - 1} = \sum_{n=0}^{\infty} \left(x^{n(q+2)} - x^{n(q+2)+q+1} \right)$, otherwise $D_q(x) = \frac{x^{q+1} + 1}{x^{q+2} + 1} = \sum_{n=0}^{\infty} (-1)^n \left(x^{n(q+2)} + x^{n(q+2)+q+1} \right)$. All involved coefficients are from $\{-1, 0, 1\}$, and the parity condition holds.   □

The following two corollaries are obtained by respectively calculating the expressions: $W_q(x, 1)$, $\frac{\partial W_q(x,y)}{\partial y}\big|_{y=1}$ and $\frac{\partial W_q(xy,1/y)}{\partial y}\big|_{y=1}$.

Corollary 2. *For $q \geqslant 1$, the generating function $F_q(x) = \sum_{n \geqslant 0} f_n x^n$ where the coefficient f_n is the number of n-length q-decreasing words is given by:*

$$F_q(x) = \frac{1 - x^{q+1}}{1 - 2x + x^{q+2}}.$$

Note that, as predicted by Theorem 1, $F_q(x) = B_q(x, 1)$, see Lemma 1 and the remark following it.

The *frequency* of a symbol in a set of words is the overall number of the occurrences of the symbol in the words of the set.

Corollary 3. *For $q \geqslant 1$, the generating function $P_{q,1}(x) = \sum_{n \geqslant 0} p_n x^n$ where the coefficient p_n is the frequency of 1s in all n-length q-decreasing words is*

$$P_{q,1}(x) = \frac{x \left(1 - qx^q + qx^{q+1} - 2x^{q+1} + x^{2q+2}\right)}{\left(1 - 2x + x^{q+2}\right)^2}.$$

Similarly, the generating function for the frequency of 0s in all n-length q-decreasing words is

$$P_{q,0}(x) = \frac{x \left(1 - x^q\right)}{\left(1 - 2x + x^{q+2}\right)^2}.$$

The frequency of 1s in $\mathcal{B}_n(11)$ is equal to the number of edges in the Fibonacci cube [13] of order n, see [14] and comments to the sequence A001629 in [26]. The generating function $P_{1,0}(x)$ allows us to show that the frequency of 0s in $\mathcal{W}_n^1$ is a shift of the sequence A006478 enumerating the number of edges in the *Fibonacci hypercube* [21], i.e., in a polytope determined by the convex hull of the Fibonacci cube.

Despite the q-decreasing words and Fibonacci words have quite different definitions, they are equinumerous and share some common features. We end this section by showing that the relative frequency of 1s (defined formally below) in both sets have the same limit when n tends to infinity.

If ω_n (resp. β_n) is the ratio between the frequency of 1s and that of 0s in the words in $\mathcal{W}_n^1$ (resp. in $\mathcal{B}_n(11)$), then $\lim_{n \to \infty} \omega_n = \lim_{n \to \infty} \beta_n$. Indeed, extracting the coefficients of x^n in both $P_{1,1}$ and of $P_{1,0}$, their ratio tends to $2 - \varphi \approx 0.3819660113$ when n tends to infinity, where φ is the golden ratio; and this is also the limit of β_n, which is obtained by investigating the ratio of the coefficients of x^n in $\frac{\partial B_q(x,y)}{\partial y}\big|_{y=1}$ and in $\frac{\partial B_q(xy,1/y)}{\partial y}\big|_{y=1}$, where $B_q(x, y)$ is from Lemma 1.

The *relative frequency of 1s* in a set of binary words is the ratio between the (cumulative) frequency of 1s and the overall number of bits in the words of the set. Alternatively, it is the expected value when a bit is randomly chosen in the words of the set. With the notations above, we have that the relative frequency of 1s in $\mathcal{W}_n^1$ is $\frac{1}{1+\frac{1}{\omega_n}}$ and in $\mathcal{B}_n(11)$ is $\frac{1}{1+\frac{1}{\beta_n}}$, and we have the next result.

Corollary 4. *The relative frequency of 1s in $\mathcal{W}_n^1$ and in $\mathcal{B}_n(11)$ both tend to $\frac{2-\varphi}{3-\varphi}$ when n tends to infinity, where φ is $\frac{1+\sqrt{5}}{2}$.*

More generally, for any $q \geqslant 1$, the overall number of bits in both sets $\mathcal{W}_n^q$ and $\mathcal{B}_n(1^{q+1})$ is $n \cdot f_{n+q+1,q+1}$, and due to the second rule in relation (4) defining the bijection $\phi : \mathcal{B}(1^{q+1}) \to \mathcal{W}^q$ we have that in $\mathcal{W}_n^q$ there are more 1s than in

$\mathcal{B}_n(1^{q+1})$. However, the next corollary shows that the difference between the relative frequency of 1s in $\mathcal{W}_n^q$ and that in $\mathcal{B}_n(1^{q+1})$ tends to zero when n tends to infinity.

Corollary 5. *For any $q \geqslant 1$, if $u_{n,q}$ (resp. $v_{n,q}$) is the frequency of 1s in $\mathcal{W}_n^q$ (resp. in $\mathcal{B}_n(1^{q+1})$), then we have*

$$\lim_{n \to \infty} \frac{u_{n,q} - v_{n,q}}{n \cdot f_{n+q+1,q+1}} = 0.$$

Proof. Since, for any $q \geqslant 1$, $u_{n,q} - v_{n,q} \geqslant 0$, it suffices to prove that we have $u_{n,q} - v_{n,q} \leqslant f_{n+q+1,q+1}$. Using Corollary 3 and Lemma 1, the generating function $H(x)$ where the coefficient of x^n is $f_{n+q+1,q+1} + v_{n,q+1} - u_{n,q}$ is

$$H(x) = B_q(x,1) + \left. \frac{\partial B_q(x,y)}{\partial y} \right|_{y=1} - P_{q,1}(x)$$

$$= \frac{1 - 2x^{q+1}}{1 - 2x + x^{q+2}},$$

which satisfies the functional equation $H(x) = 1 - 2x^{q+1} + 2xH(x) - x^{q+2}H(x)$. By a simple observation, $H(x)$ is also the generating function with respect to the length of binary words different from 0^{q+1} and 1^{q+1} and that do not start with 0^{q+2}. Then we have $u_{n,q} - v_{n,q} \leqslant f_{n+q+1,q+1}$. Dividing by $nf_{n+q+1,q+1}$, and taking the limit when n tends to infinity, we obtain the expected result. □

Corollary 6. *For $q \geqslant 1$ the relative frequency of 1s in $\mathcal{W}_n^q$ and in $\mathcal{B}_n(1^{q+1})$ have a common non-zero limit when n tends to infinity.*

Proof. Corollary 4 says that, for $q = 1$, the relative frequency of 1s in $\mathcal{W}_n^q$ and in $\mathcal{B}_n(1^{q+1})$ have a common limit when n tends to infinity. For $q \geqslant 2$, Corollary 5 does not ensure that each of the relative frequency of 1s in $\mathcal{W}_n^q$ (that is $\frac{u_{n,q}}{n \cdot f_{n+q+1,q+1}}$) and in $\mathcal{B}_n(1^{q+1})$ (that is $\frac{v_{n,q}}{n \cdot f_{n+q+1,q+1}}$) has a limit when n tends to infinity. However, in [2], using asymptotic analysis it is shown that $\frac{v_{n,q}}{n \cdot f_{n+q+1,q+1}}$ converges to a non-zero value when n tends to infinity, and the limit can be approximated by numerical methods. From Corollary 5 it follows that so does $\frac{u_{n,q}}{n \cdot f_{n+q+1,q+1}}$, and the two limits are equal. □

4 Exhaustive generation and Gray codes for *q*-decreasing words

Here we show that q-decreasing words can be efficiently generated in lexicographic order and explain how the obtained generating algorithm can be turned into a 3-Gray code generating algorithm. Then, we give a more intricate construction of a 1-Gray code for the particular case $q = 1$. As a byproduct, this construction gives a positive answer for the existence a Hamiltonian path in Fibonacci-run graph conjectured in [9].

4.1   3-Gray codes and exhaustive generation

Algorithm in Figure 1 generates prefixes of q-decreasing words in the lexicographical order, and eventually all n-length q-decreasing words. The size n, parameter q and the array w of length $n + 1$ are global variables and the main call is LexFib$(1, n)$. For convenience, $w[0]$ is initialized by 1 and the parameter *delta* is the number of consecutive 1s that can be added to the current generated prefix without violating the q-decreasingness.

It can be seen that this recursive algorithm satisfies Baronaigien and Ruskey's *constant amortized time (CAT) principle* in [4] stated below. As noticed in [4], by considering the underlying computation tree it follows that any algorithm satisfying the CAT principle is efficient, and thus LexFib is an efficient generating algorithm. We refer the reader to [23, Section 1.7] for more about CAT generating algorithms.

CAT principle [4]: A recursive generation algorithm with the following properties runs in constant amortized time. (i) Every call results in the output of at least one object; (ii) Excluding the computation done by recursive calls the amount of computation of any call is proportional to the degree of the call (that is, the number of subsequent recursive calls produced by the call); (iii) The number of calls of degree one is $O(N)$, where N is the number of generated objects.

```
procedure LexFib(pos, delta: integer)
   if (pos = n + 1) print w;
   else  w[pos] := 0;
         if (w[pos − 1] = 1) d := q − 1; else d:=delta + q; endif
         LexFib(pos + 1, d);
         if (delta > 0)
             w[pos] := 1; LexFib(pos + 1, delta − 1);
         endif
   endif
end procedure
```

Fig. 1: Lexicographic generation algorithm for q-decreasing words.

Generally, the bijection ϕ in (4) does not preserve *Graycodeness*, i.e., a Gray code for $(q + 1)$-Fibonacci words is not necessarily mapped by ϕ to a Gray code for q-decreasing words. For instance, when $q = 1$ and $n = 2k + 1$ the Gray code for Fibonacci words in [28] always contains two consecutive words $u = (10)^k 1$ and $v = (10)^k 0$, but their images $\phi(u) = 1^{2k+1}$ and $\phi(v) = 0^{k+1} 1^k$ have arbitrarily large Hamming distance for enough large n. A similar phenomenon happens when $n = 2k$ and $q = 1$ with $u = (10)^{k-2} 10$ and $v = (10)^{k-2} 00$: $\phi(u) = 1^{2k}$ and $\phi(v) = 0^{k+1} 1^{k-1}$. See also Table 1(a) for the image through ϕ of the 1-Gray code in [28] for $\mathcal{B}_4(111)$.

$u \in \mathcal{B}_4(111)$	$\phi(u) \in \mathcal{W}_4^2$
1100	0011
1101	1111
1001	1001
1000	0001
1010	0101
1011	1101
0011	1100
0010	0100
0000	0000
0001	1000
0101	1010
0100	0010
0110	1110

(a)

Words in $\mathcal{W}_6^1$ in BRGC order

1	000000		*12*	111100	3
2	000001	1	*13*	111111	2
3	000011	1	*14*	111110	1
4	000010	1	*15*	111001	3
5	000110	1	*16*	111000	1
6	000100	1	*17*	100100	3
7	001001	3	*18*	100010	2
8	001000	1	*19*	100011	1
9	110000	3	*20*	100001	1
10	110001	1	*21*	100000	1
11	110010	2			

(b)

Table 1: (a) The images of words in $\mathcal{B}_4(111)$ under the bijection ϕ. Words in $\mathcal{B}_4(111)$ are listed in a BRGC-like order, called local reflected order in [28], which yields a 1-Gray code order. (b) The set $\mathcal{W}_6^1$ in BRGC order together with the Hamming distance between consecutive words.

Below we show that BRGC order (that is, the order induced by Binary Reflected Gray Code in [12]) yields a 3-Gray code on $\mathcal{W}_n^q$. Much more interestingly, thanks to Corollary 1, the necessary condition for the existence of a 1-Gray code is satisfied, and we will provide such a Gray code for $\mathcal{W}_n^1$ in the following part.

In [29] the author introduces the notion of *absorbent set*, which (up to complement) is defined as: a binary word set $\mathcal{X} \subset \{0,1\}^n$ is called absorbent if for any $u \in \mathcal{X}$ and any k, $1 \leqslant k < n$, $u_1 u_2 \ldots u_k 0^{n-k}$ is also a word in $\mathcal{X}$. Corollary 1 from the same paper proves that when an absorbent set $\mathcal{X}$ is listed in BRGC order (that is, restricting BRGC to $\mathcal{X}$) the resulting listing yields a 3-Gray code. Clearly, $\mathcal{W}_n^q$ is an absorbent set and we have the following consequence.

Corollary 7. *For $n \geqslant 0, q \geqslant 1$, the restriction of BRGC order yields a 3-Gray code for $\mathcal{W}_n^q$.*

See for an example Table 1(b).

Applying reversing sublists technique [22] by adding a new parameter to LexFib which keeps track of the parity of the number of 1s in the current generated prefix, procedure LexFib can be turned into one generating the same class of words in BRGC order instead of the lexicographic one, and so according to Corollary 7 producing a 3-Gray code for $\mathcal{W}_n^q$. Obviously the obtained Gray code generating algorithm inherits CAT property.

At this point it is worth mentioning an alternative powerful framework to design Gray codes as sublists of BRGC. In [24] it is given the following definition (up to mirroring each word): a set of same length binary words is a *flip-swap language* (with respect to 1) if it is closed under the operations (i) flip the rightmost 1, and (ii) swap the rightmost 1 with the bit to its left. Theorem 2 in [24] states that when a flip-swap language is listed in BRGC order the resulting listing yields a 2-Gray code, and in the same paper there is given a variety of combinatorial classes that are flip-swap languages together with efficient generating algorithms for most of them. It is easy to see that absorbent sets coincide with languages closed under flip operation, and thus it is not surprising that, when listed in BRGC order, flip-swap languages yield more restrictive (that is, 2- instead of 3-) Gray codes than absorbent sets do. In particular, for $n \geqslant 0$ and $q \geqslant 1$, $\mathcal{W}_n^q$ is an absorbent set (or equivalently, a flip-closed language) but not a flip-swap language.

4.2   1-Gray code for $\mathcal{W}_n^1$

In this section, we construct a 1-Gray code for the set $\mathcal{W}_n^q$, $n \geqslant 0$, when $q = 1$, which in particular gives a positive answer to a conjecture in [9]. For this purpose, we decompose $\mathcal{W}_n^1$ $n \geqslant 1$, as

$$\mathcal{W}_n^1 = \mathcal{Z}_n \cup 1 \cdot \mathcal{W}_{n-1}^1$$

where $\mathcal{W}_0^1$ consists of the empty word ϵ, and $\mathcal{Z}_n$ is the subset of words starting with 0 in $\mathcal{W}_n^1$. In turn, we decompose $\mathcal{Z}_n$ as

$$\mathcal{Z}_n = \{0^n\} \cup \bigcup_{r=3}^{n} \mathcal{D}_n^r$$

where $\mathcal{D}_n^r = \bigcup_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} 0^{r-j} 1^j \cdot \mathcal{Z}_{n-r}$. We refer to Figure 2(a) for a graphical illustration of the decomposition of $\mathcal{Z}_n$ for $n = 17$ where the point at coordinates (i, j) corresponds to the set $0^i 1^j \cdot \mathcal{Z}_{n-i-j}$, $1 \leqslant j < i \leqslant n - 1$, $3 \leqslant i + j \leqslant n$, except the lowest point which corresponds to $\{0^n\}$. The sets $\mathcal{D}_n^r$, $3 \leqslant r \leqslant n$, correspond to the southwest-northeast diagonals of the graphic.

Let $\mathbf{Z}_0 = \epsilon$ be the list containing only the empty word ϵ, $\mathbf{Z}_1 = 0, \mathbf{Z}_2 = 00, \mathbf{Z}_3 = 000 \circ 001$. If $\mathbf{L}$ is a list and w is a word, then $w \cdot \mathbf{L}$ denotes the list where w is concatenated to every word from $\mathbf{L}$. According to the above definitions, it is straightforward to check the following lemma.

Lemma 3. *For any $1 \leqslant k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with* $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ *and* $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$*. Given i and j such that $1 \leqslant j < i \leqslant n$, then*

(i) for $3 \leqslant i + j < n$, the list $\mathbf{L} = 0^i 1^j \cdot \mathbf{Z}_{n-i-j}$ is a 1-Gray code for $0^i 1^j \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i 1^j 0(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^i 1^j (001)^\star$;
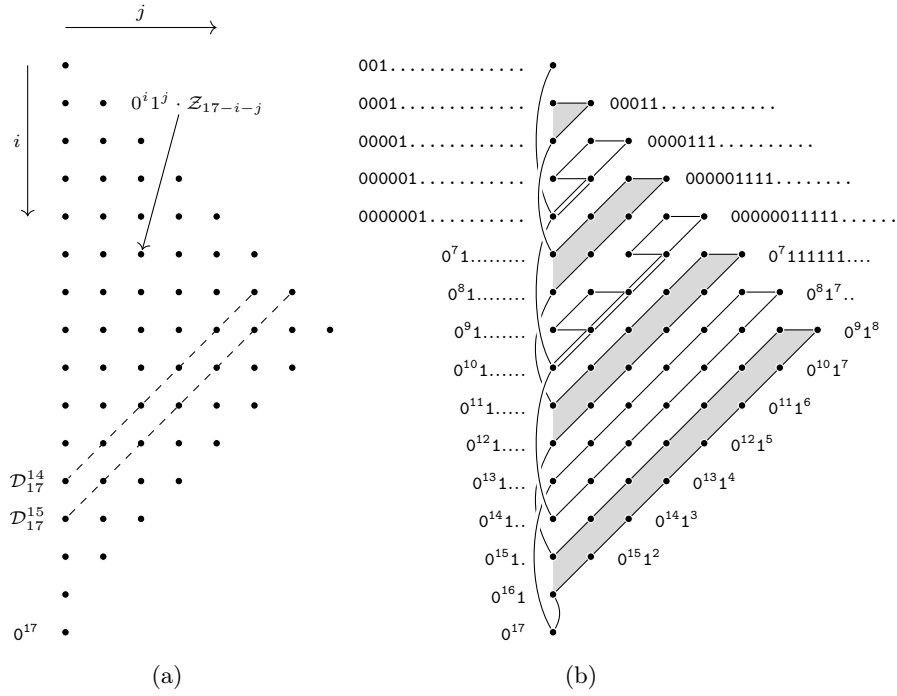*when $i + j = n$ the list $\mathbf{L}$ contains only the word $0^i 1^j$;*

Fig. 2: (a) A decomposition of $\mathcal{Z}_{17}$ as a union of subsets $0^i 1^j \cdot \mathcal{Z}_{17-i-j}$ (or equivalently a union of diagonals $\mathcal{D}_{17}^r$) and $\{0^{17}\}$. (b) An illustration of the 1-Gray code $\mathbf{Z}_{17}$. The pairs of consecutive diagonals dealt with Lemma 3 are shown in gray-filled area; the other pairs are dealt with Lemma 4. A point labelled $0^9 1$....... (that is $0^9 1$ followed by seven dots) corresponds to the set of words in $0^9 1 \cdot \mathcal{Z}_7$.

(ii) for $3 \leqslant i+j < n-1$, the list $\mathbf{L} = 0^i 1^{j+1} \cdot \mathbf{Z}_{n-i-j-1} \circ 0^i 1^j \cdot \mathbf{Z}_{n-i-j}$ is a 1-Gray code for $0^i 1^{j+1} \cdot \mathcal{Z}_{n-i-j-1} \cup 0^i 1^j \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i 1^{j+1} 0(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^i 1^j (001)^\star$;
   when $i+j+1 = n$, the list $\mathbf{L} = 0^i 1^{j+1} \circ 0^i 1^j 0$;
(iii) for $3 \leqslant i+j < n$, the list $\mathbf{L} = 0^i 1^j \cdot \mathbf{Z}_{n-i-j} \circ \overleftarrow{0^{i-1} 1^{j+1} \cdot \mathbf{Z}_{n-i-j}}$ is a 1-Gray code for $0^i 1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^{i-1} 1^{j+1} \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i 1^j 0(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^{i-1} 1^{j+1} 0(001)^\star$;
   when $i+j = n$, the list $\mathbf{L} = 0^i 1^j \circ 0^{i-1} 1^{j+1}$;
(iv) for $3 \leqslant i+j \leqslant n$, the list $\mathbf{L} = \overleftarrow{0^i 1^j \cdot \mathbf{Z}_{n-i-j}} \circ 0^{i-1} 1^{j+1} \cdot \mathbf{Z}_{n-i-j}$ is a 1-Gray code for $0^i 1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^{i-1} 1^{j+1} \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i 1^j (001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^{i-1} 1^{j+1}(001)^\star$.

Lemma 4. *Let us consider $r = 1 \mod 4$, $3 \leqslant r \leqslant n$. For any $1 \leqslant k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ and $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$.*

(i) If $r \neq n-1$, then there is a 1-Gray code Δ_n^r for $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ such that first$(\Delta_n^r) = 0^{r-2}1(001)^\star$ and last$(\Delta_n^r) = 0^{r-1}1(001)^\star$.

(ii) If $r = n-1$, then there is a 1-Gray code Δ_n^{n-1} for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2}$ such that first$(\Delta_n^{n-1}) = 0^{n-3}100$ and last$(\Delta_n^{n-1}) = 0^{n-1}1$.

Proof. For the first assertion (i), it suffices to consider the list

$$\Delta_n^r = \bigcirc_{j=1}^{\frac{r-3}{2}} 0^{r-1-j}1^j \cdot \langle \mathbf{Z}_{n-r+1}\rangle^j \circ \overleftarrow{\bigcirc_{j=1}^{\frac{r-1}{2}} 0^{r-j}1^j \cdot \langle \mathbf{Z}_{n-r}\rangle^j}.$$

After considering assertions of Lemma 3, it remains to examine the transition between $w_0 = \mathrm{last}(0^{r-1-j_0}1^{j_0} \cdot \langle \mathbf{Z}_{n-r+1}\rangle^{j_0})$ for $j_0 = \frac{r-3}{2}$ and $w_1 = \mathrm{first}(\overleftarrow{0^{r-j_1}1^{j_1} \cdot \langle \mathbf{Z}_{n-r}\rangle^{j_1}})$ for $j_1 = \frac{r-1}{2}$. Since $r = 1 \mod 4$, we have necessarily j_0 is odd and $j_0 + 1 = j_1$ which implies that $w_0 = \mathrm{last}(0^{r-1-j_0}1^{j_0} \cdot \overleftarrow{\mathbf{Z}_{n-r+1}}) = 0^{r-1-j_0}1^{j_0}0(001)^\star$ and $w_1 = \mathrm{first}(\overleftarrow{0^{r-j_1}1^{j_1} \cdot \mathbf{Z}_{n-r}}) = 0^{r-j_1}1^{j_1}(001)^\star$, and they differ by exactly one bit.

For the second assertion (ii), we consider the list

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1^j 00 \circ 0^{\frac{n}{2}}1^{\frac{n-2}{2}}0 \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-4}{2}} \langle 0^{n-j-1}1^{j+1} \circ 0^{n-1-j}1^j 0\rangle^{j-1}} \circ 0^{n-1}1.$$

A simple study of each kind of transitions allows us to see that Δ_n^{n-1} is a 1-Gray code for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2}$ satisfying first$(\Delta_n^{n-1}) = 0^{n-3}100$ and last$(\Delta_n^{n-1}) = 0^{n-1}1$. An illustration of this Gray code for $n = 10$ (and thus $r = 9$) can be found in the last sketch of Figure 5. $\quad\square$

In the following, we write $w\mathbf{L}$ instead of $w \cdot \mathbf{L}$ to be more concise.

Lemma 5. *Let us consider* $r = 3 \mod 4$, $3 < n$ *and* $3 \leqslant r \leqslant n$. *For any* $1 \leqslant k < n$, *we suppose that* $\mathbf{Z}_k$ *is a 1-Gray code for* $\mathscr{Z}_k$ *with* first$(\mathbf{Z}_k) = 0(001)^\star$ *and* last$(\mathbf{Z}_k) = (001)^\star$.

(i) *If* $r = 3$, *then there is a 1-Gray code* Δ_n^3 *for* $\mathcal{D}_n^3$ *such that* first$(\Delta_n^3) = 0010(001)^\star$ *and* last$(\Delta_n^3) = (001)^\star$.

(ii) *If* $r = n-2$, *then there is a 1-Gray code* Δ_n^{n-2} *for* $\mathcal{D}_n^{n-2} \cup \mathcal{D}_n^{n-3}$ *such that* first$(\Delta_n^{n-2}) = 0^{n-4}1000$ *and* last$(\Delta_n^{n-2}) = 0^{n-3}100$.

(iii) *If* $r = n-1$, *then there is a 1-Gray code* Δ_n^{n-1} *for* $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2} \backslash \{0^{n-1}1\}$ *such that* first$(\Delta_n^{n-1}) = 0^{n-3}100$ *and* last$(\Delta_n^{n-1}) = 0^{n-2}10$.

(iv) *If* $r = n$, *then there is a 1-Gray code* Δ_n^n *for* $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1}$ *such that* first$(\Delta_n^n) = 0^{n-2}10$ *and* last$(\Delta_n^n) = 0^{n-1}1$.

(v) *If* $r \notin \{3, n-2, n-1, n\}$, *then there is a 1-Gray code* Δ_n^r *for* $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ *such that* first$(\Delta_n^r) = 0^{r-1}10(001)^\star$ *and* last$(\Delta_n^r) = 0^{r-1}1(001)^\star$.

Proof. For the case (i), we set: $\Delta_3 = 0^2 1\mathbf{Z}_{n-3}$.

For the case (ii), we set: $\Delta_n^{n-2} = \bigcirc_{j=1}^{\frac{n-5}{2}} 0^{n-3-j}1^j \langle \mathbf{Z}_3\rangle^{j-1} \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-3}{2}} 0^{n-2-j}1^j \mathbf{Z}_2}$. Since we have $\mathbf{Z}_2 = 00$ and $\mathbf{Z}_3 = 000 \circ 001$, it is straightforward to see that Δ_n^{n-2} is a 1-Gray code with first$(\Delta_n^{n-2}) = 0^{n-4}1000$ and last$(\Delta_n^{n-2}) = 0^{n-3}100$.

For the case (iii), we set:

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1^j \mathbf{Z}_2 \circ 0^{\frac{n}{2}}1^{\frac{n-2}{2}}\mathbf{Z}_1 \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-4}{2}} \left\langle 0^{n-1-j}1^{j+1} \circ 0^{n-1-j}1^j\mathbf{Z}_1 \right\rangle^j}.$$

Knowing that $\mathbf{Z}_2 = 00$ and $\mathbf{Z}_1 = 0$, we can easily check that any pair of consecutive words differ by exactly one bit, which proves that Δ_n^{n-1} is a 1-Gray code.

For the case (iv), we set: $\Delta_n^n = \bigcirc_{j=1}^{\frac{n-3}{2}} 0^{n-1-j}1^j 0 \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-1}{2}} 0^{n-j}1^j}$. As previously the result can be obtained easily.

The case (v) is more challenging to handle. The set $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ consists of the union of the following subsets: $K_1 = 0^{r-2}1\mathcal{Z}_{n-r+1}$, $K_2 = 0^{r-3}11\mathcal{Z}_{n-r+1}$, …, $K_a = 0^{r-a-1}1^a\mathcal{Z}_{n-r+1}$ and $L_1 = 0^{r-1}1\mathcal{Z}_{n-r}$, $L_2 = 0^{r-2}11\mathcal{Z}_{n-r}$, …, $L_{a+1} = 0^{r-a-1}1^{a+1}\mathcal{Z}_{n-r}$ with $a = \lfloor\frac{r-2}{2}\rfloor = \frac{r-3}{2}$. Let us denote by $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_a$ and $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_{a+1}$ the associated Gray codes obtained by replacing $\mathcal{Z}_k$ with the Gray code $\mathbf{Z}_k$.

Remark that for $1 \leqslant i \leqslant a-1$ (resp. $1 \leqslant i \leqslant a$) and for a given j, the jth word of $\mathbf{K}_i$ (resp. $\mathbf{L}_i$) and the jth word of $\mathbf{K}_{i+1}$ (resp. $\mathbf{L}_{i+1}$) differ by exactly one bit. Recall that $\widehat{\mathbf{A}}$ is obtained from a list $\mathbf{A}$ by deleting its first element. The words $\mathrm{first}(\mathbf{K}_i)$ and $\mathrm{last}(\mathbf{L}_{i+1})$ differ by one bit. We have $\mathrm{first}(\widehat{\mathbf{L}_{a+1} \circ \mathbf{K}_a}) = \mathrm{first}(\widehat{\mathbf{L}_{a+1}})$ and it differs obviously by one bit from $\mathrm{first}(\mathbf{L}_{a+1})$. We also have $r = 3 \mod 4$, so $\frac{r-3}{2} = a$ is even. Taking into account all these remarks, the list Δ_n^r defined below is a Gray code:

$$\Delta_n^r = \bigcirc_{i=1}^{a+1} \mathrm{first}(\mathbf{L}_i) \circ \bigcirc_{i=a}^1 \left\langle \widehat{\mathbf{L}_{i+1}} \circ \mathbf{K}_i \right\rangle^i \circ \widehat{\mathbf{L}_1}$$

We refer to Figure 3 for a graphical representation of this Gray code. □

Theorem 3. *For any $n \geqslant 0$, there exists a 1-Gray code $\mathbf{Z}_n$ for $\mathcal{Z}_n$ such that* $\mathrm{first}(\mathbf{Z}_n) = 0(001)^\star$ *and* $\mathrm{last}(\mathbf{Z}_n) = (001)^\star$.

Using initial conditions $\mathbf{Z}_0 = \epsilon, \mathbf{Z}_1 = 0, \mathbf{Z}_2 = 00, \mathbf{Z}_3 = 000 \circ 001$, where ϵ is the empty word, and the recursive constructions for lists Δ_n^r (Lemmas 3 and 4) we define the 1-Gray code $\mathbf{Z}_n$ as follows:

$$\mathbf{Z}_n = \begin{cases} \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^n \circ 0^n \circ \Delta_n^{n-2} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 1 \mod 4, \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-1} \circ 0^n \circ \Delta_n^{n-3} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 2 \mod 4, \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-2} \circ 0^n \circ \Delta_n^n \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 3 \mod 4. \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-3} \circ 0^{n-1}1 \circ 0^n \circ \Delta_n^{n-1} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 0 \mod 4. \end{cases}$$

Due to Lemmas 4 and 5, $\mathrm{last}(\Delta_n^{4i+1})$ differs by one bit from $\mathrm{first}(\Delta_n^{4i+5})$ and $\mathrm{first}(\Delta_n^{4i+3})$ differs by one bit from $\mathrm{last}(\Delta_n^{4i+7})$. Using 0^n (and $0^{n-1}1$ when $n = 0 \mod 4$) we connected all Δ_n^r and ensure that $\mathbf{Z}_n$ is a 1-Gray code. See Table 2 for a more detailed view on the structure of $\mathbf{Z}_n$. □

We refer to Figure 5 for a graphical representation of $\mathbf{Z}_n$ for $4 \leqslant n \leqslant 10$, see also Figure 2(b) for $n = 17$. An immediate consequence of Theorem 3 is the following.
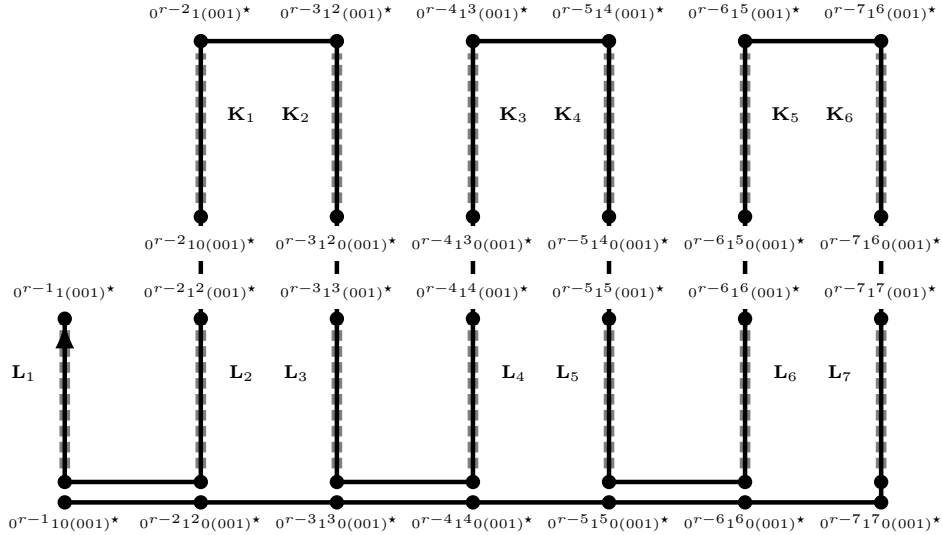
Fig. 3: An illustration of the Gray code Δ_n^r for the case (v) in the proof of Lemma 4 (we consider $a = 6$, $r = 15$). Vertical sequences of squares are Gray codes $\mathbf{K}_i$, $1 \leqslant i \leqslant a$, and $\mathbf{L}_i$, $1 \leqslant i \leqslant a + 1$, so that the first and the last elements are respectively the bottom and top squares of the segments. The walk illustrates the Gray code Δ_n^r that starts with $\mathrm{first}(\mathbf{L}_1)$ and ends with $\mathrm{last}(\mathbf{L}_1)$.

Theorem 4. *For any $n \geqslant 1$, $\mathbf{W}_n^1 = 1 \cdot \mathbf{W}_{n-1}^1 \circ \mathbf{Z}_n$ is a 1-Gray code for $\mathcal{W}_n^1$ such that $\mathrm{first}(\mathbf{W}_n^1) = 1^n$, $\mathrm{last}(\mathbf{W}_n^1) = (001)^\star$, and where $\mathbf{W}_0^1$ is a list containing only the empty word.*

However, the efficient generation of this Gray code remains an open problem.

Eğecioğlu and Iršič introduce in [9] the *"run-constrained binary strings"*. These are binary words, in which every run of 1s is immediately followed by a strictly longer run of 0s. Using these strings of length $n+2$ as vertices, and connecting two vertices if they differ at only one position, the authors of [9] form the *Fibonacci-run graph* $\mathcal{R}_n$ as the induced subgraph of the hypercube. (As every non-empty run-constrained string must end with 00, authors of [9] actually drop the last 2 zeros, but we do not.) Figure 4 gives small examples.

It turns out that the run-constrained binary strings are precisely the reverse of 1-decreasing words beginning with 0. In this light, the Gray code $\mathbf{Z}_n$ in Theorem 3 gives a Hamiltonian path in the Fibonacci-run graph. The next corollary settles a conjecture in [9].

Corollary 8. *For any $n \geqslant 1$, the Fibonacci-run graph $\mathcal{R}_n$ has a Hamiltonian path.*

Lemma 9.1 from [9] says that if $n \neq 1 \mod 3$, then $\mathcal{R}_n$ does not contain a Hamiltonian cycle. Our method gives a Hamiltonian path, which is not a cycle.

	$n = 1 \bmod 4$		$n = 2 \bmod 4$		$n = 3 \bmod 4$		$n = 0 \bmod 4$	
Lemma 4	$\Delta_n^5 =$	0001(001)* $\ldots$ 00001(001)*	$\Delta_n^5 =$	0001(001)* $\ldots$ 00001(001)*	$\Delta_n^5 =$	0001(001)* $\ldots$ 00001(001)*	$\Delta_n^5 =$	0001(001)* $\ldots$ 00001(001)*
	$\Delta_n^9 =$	00000 001(001)* $\ldots$ 000000001(001)*	$\Delta_n^9 =$	00000 001(001)* $\ldots$ 000000001(001)*	$\Delta_n^9 =$	00000 001(001)* $\ldots$ 000000001(001)*	$\Delta_n^9 =$	00000 001(001)* $\ldots$ 000000001(001)*
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
	$\Delta_n^{n-4} =$	$0^{n-6}1(001)^\star$ $\ldots$ $0^{n-5}1(001)^\star$	$\Delta_n^{n-5} =$	$0^{n-7}1(001)^\star$ $\ldots$ $0^{n-6}1(001)^\star$	$\Delta_n^{n-6} =$	$0^{n-8}1(001)^\star$ $\ldots$ $0^{n-7}1(001)^\star$	$\Delta_n^{n-7} =$	$0^{n-9}1(001)^\star$ $\ldots$ $0^{n-8}1(001)^\star$
	$\Delta_n^n =$	$0^{n-2}10$ $\ldots$ $0^{n-1}1$	$\Delta_n^{n-1} =$	$0^{n-3}100$ $\ldots$ $0^{n-1}1$	$\Delta_n^{n-2} =$	$0^{n-4}1(001)^\star$ $\ldots$ $0^{n-3}1(001)^\star$	$\Delta_n^{n-3} =$	$0^{n-5}1(001)^\star$ $\ldots$ $0^{n-4}1(001)^\star$
		0^n		0^n		0^n		$0^{n-1}1$ 0^n
Lemma 5	$\Delta_n^{n-2} =$	$0^{n-4}1000$ $\ldots$ $0^{n-3}100$	$\Delta_n^{n-3} =$	$0^{n-4}10(001)^\star$ $\ldots$ $0^{n-4}1(001)^\star$	$\Delta_n^n =$	$0^{n-2}10$ $\ldots$ $0^{n-1}1$	$\Delta_n^{n-1} =$	$0^{n-3}100$ $\ldots$ $0^{n-2}10$
	$\Delta_n^{n-6} =$	$0^{n-7}10(001)^\star$ $\ldots$ $0^{n-7}1(001)^\star$	$\Delta_n^{n-7} =$	$0^{n-8}10(001)^\star$ $\ldots$ $0^{n-8}1(001)^\star$	$\Delta_n^{n-4} =$	$0^{n-5}10(001)^\star$ $\ldots$ $0^{n-5}1(001)^\star$	$\Delta_n^{n-5} =$	$0^{n-6}10(001)^\star$ $\ldots$ $0^{n-6}1(001)^\star$
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
	$\Delta_n^7 =$	00000010(001)* $\ldots$ 0000 001(001)*	$\Delta_n^7 =$	00000010(001)* $\ldots$ 0000 001(001)*	$\Delta_n^7 =$	00000010(001)* $\ldots$ 0000 001(001)*	$\Delta_n^7 =$	00000010(001)* $\ldots$ 0000 001(001)*
	$\Delta_n^3 =$	0010(001)* $\ldots$ (001)*	$\Delta_n^3 =$	0010(001)* $\ldots$ (001)*	$\Delta_n^3 =$	0010(001)* $\ldots$ (001)*	$\Delta_n^3 =$	0010(001)* $\ldots$ (001)*

Table 2: The structure of $\mathbf{Z}_n$.

The question of whether there is a Hamiltonian cycle for the case $n = 1 \bmod 3$ remains open.

Finally, the validity of the parity condition stated in Corollary 1 and experimental investigations for small values, $0 \leqslant n \leqslant 5$ and $2 \leqslant q \leqslant 5$, suggest the following extension of Theorem 4.

Conjecture 1 *For any $n \geqslant 0$ and $q \geqslant 1$, there is a 1-Gray code for $\mathcal{W}_n^q$.*

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1	111111	*8*	110010	*15*	000110
2	111110	*9*	100010	*16*	000010
3	111100	*10*	100011	*17*	000011
4	111000	*11*	100001	*18*	000001
5	111001	*12*	100000	*19*	000000
6	110001	*13*	100100	*20*	001000
7	110000	*14*	000100	*21*	001001

Table 3: The Gray code $\mathbf{W}_6^1$ for the set $\mathcal{W}_6^1$. The Hamming distance between two consecutive words is one.
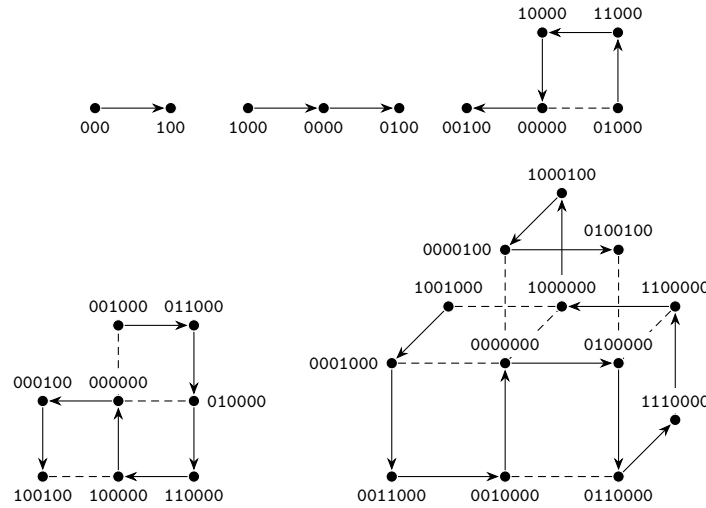
Fig. 4: Fibonacci-run graphs for small values of n. Vertices correspond to the reverse of words from $\mathcal{W}_n^1$ beginning with 0. The Hamiltonian path is provided by Corollary 8. If we read words from right to left, the path starts at $0(001)^\star$ and ends at $(001)^\star$.

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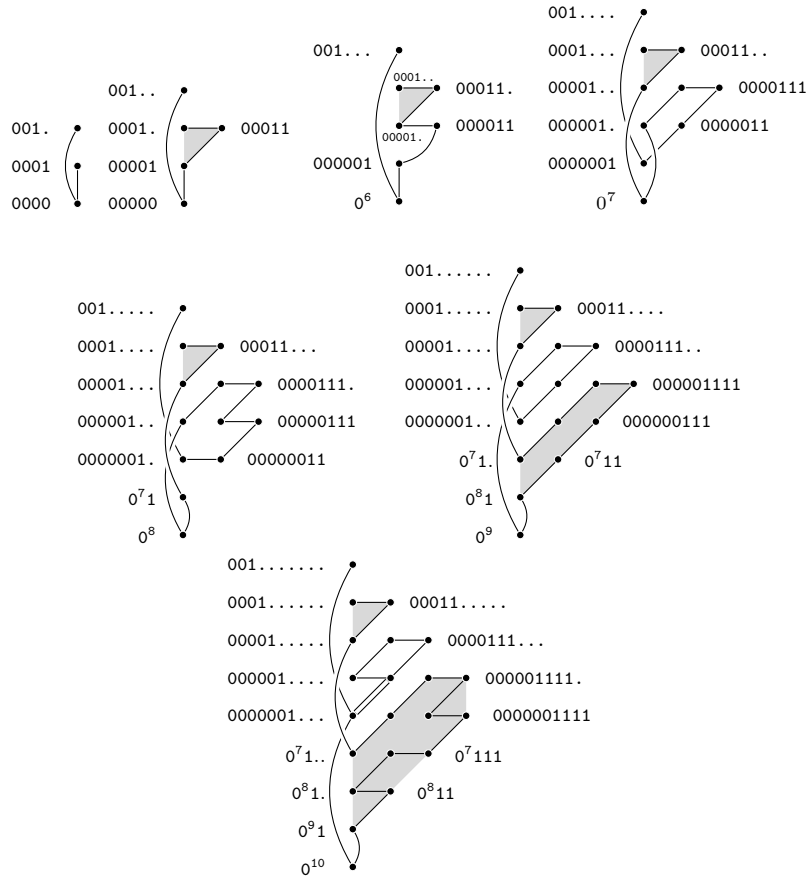
Fig. 5: An illustration of the recursive definition for the Gray codes $\mathbf{Z}_n$, $4 \leqslant n \leqslant 10$. A point labelled 001....... (that is 0001 followed by seven dots) corresponds to the set of words in $001 \cdot \mathcal{Z}_7$.

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