Qubonacci words

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(This talk is based on joint work with Jean-Luc Baril and Vincent Vajnovszki.)

An n-length binary word is *q-decreasing*, $q \in \mathbb{N}^+$, if every of its length maximal factor of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$. The set of q-decreasing words of length n is denoted by $\mathcal{W}_n^q$. For example we have

$\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$,
$\mathcal{W}_4^2 = \{0000, 0001, 0010, 0011, 0100, 0101, 1000, 1001, 1010, 1100, 1101, 1110, 1111\}$.

Denote by $\mathcal{B}_n(1^k), k \geq 2$, the set of all n-length binary words containing no occurrences of factor 1^k. The fact that $\mathcal{B}_n(1^k)$ is enumerated by k-generalized Fibonacci numbers is widely known [3, p. 286]. For $q \geq 1$ we construct a bijection between $\mathcal{B}_n(1^{q+1})$ and $\mathcal{W}_n^q$, and give bivariate generating functions according to the length of the words and the number of 1s. In general, there are more 1s in $\mathcal{W}_n^q$ than in $\mathcal{B}_n(1^{q+1})$.

The *1s frequency* of a set is the ratio between the total number of 1s and the overall number of bits in the words of the set. Alternatively, it is the expected value of a random bit in a random word of the set. It can be shown, for $q \geq 1$, that the 1s frequencies of $\mathcal{W}_n^q$ and of $\mathcal{B}_n(1^{q+1})$ converge, when n tends to infinity, to a same limit whose value is related to the generalized golden ratio φ_{q+1}, the root of largest modulus of the famous Fibonacci polynomial $x^{q+1} - x^q - \cdots - x - 1$. In particular, we show that the common limit of the 1s frequencies of $\mathcal{W}_n^1$ and of $\mathcal{B}_n(11)$ is $(2 - \varphi)/(3 - \varphi) \approx 0.27639$ with $\varphi = (1 + \sqrt{5})/2$ the golden ratio.

For any $q \geq 1$, we provide an efficient exhaustive generating algorithm for q-decreasing words in lexicographic order. We show the existence of a 3-Gray code and explain how a generating algorithm for this Gray code can be obtained. Moreover, we give the construction of a more restrictive 1-Gray code for 1-decreasing words, which in particular settles a conjecture about the Hamiltonicity of certain hypercube subgraphs stated recently by Eğecioğlu and Iršič [2]. We conjecture the existence of 1-Gray code for any $q \geq 2$. The extended version of this work is available on the arXiv [1].

The case $q \in \mathbb{Q}^+$ may be an interesting lane to explore.

[1] Baril, J.-L., Kirgizov, S., Vajnovszki, V. “Gray codes for Fibonacci q-decreasing words”. arXiv:2010.09505 [Math.CO] (2020)

[2] Eğecioğlu, O., Iršič, V. “Fibonacci-run graphs I: Basic properties”. Discrete Applied Mathematics, 295, 70–84 (2021)

[3] Knuth, D. “The Art of Computer Programming, Volume 3: Sorting and Searching”, 2nd ed. (1998)