Sturm meets Fibonacci in Minkowski's fractal bar

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joint work with Sergey Dovgal

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Generalized Fibonacci, k-bonacci

Initial terms: 0,...,0,0,1,

$f_{n,1} = f_{n-1,1} + f_{n-2,1}$, Fibonacci

$f_{n,2} = f_{n-1,2} + f_{n-2,2} + f_{n-3,2}$, Tribonacci

$f_{n,3} = f_{n-1,3} + f_{n-2,3} + f_{n-3,3} + f_{n-4,3}$, Tetranacci

Generalized Fibonacci numbers and associated matrices, 1960
E. P. Miles Jr.

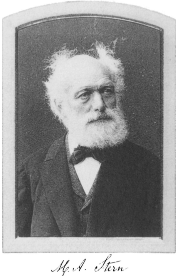
Fibonacci-Tribonacci, 1963
M. Feinberg

Can we extend
the definition of $f_{n,k}$
to cover the case where k is
not an integer?

π–bonacci numbers?

- Knuth-Fibonacci, q-decreasing, and Sturmian words
- Generalization of the golden ratio, $\Phi(q), q \in \mathbb{R}^+$
- Link to the Stern–Brocot tree and Minkowski's $?(x)$




Knuth-Fibonacci words are Binary words containing no occurrences of factor 1^k. They are enumerated by generalized Fibonacci numbers.

- Avoiding 11 : Fibonacci, $a_n = a_{n-1} + a_{n-2}$
- Avoiding 111 : Tribonacci, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$
- ...

The Art of Computer Programming, Volume 3
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```
          010
          000      Words avoiding 11
      01  001
  0   00  101  ...
  1   10  100

-----------------------------------
  2   3   5   8  .....
```

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          011
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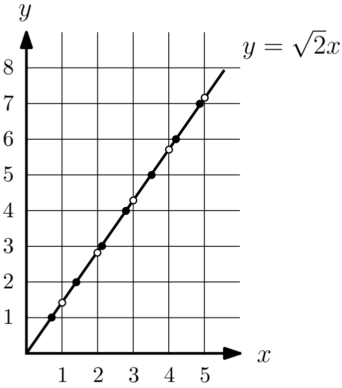
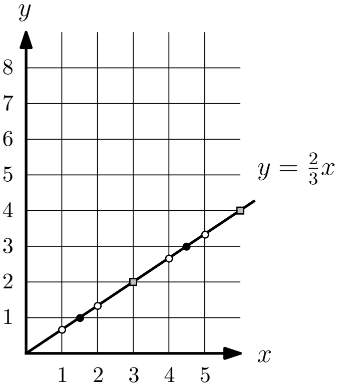
  0   00  101  ...
  1   10  100
      11  110
-----------------------------------
  2   4   7   13  .....
```

Sturmian words

Write 1 if the line intersects a horizontal edge, 0 in case of a vertical edge, 01 in case of a corner.
The resulting infinite word is $s(q)$, where $q \in \mathbb{R}^+$ is a line's slope.





$s(\sqrt{2}) = 101011010110...$ $\quad$ $s\left(\frac{2}{3}\right) = 0100101001...$

q-decreasing words

An n-length binary word is q*-decreasing*, $q \in \mathbb{R}^+$, if every of its length maximal factors of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$.

$$\cdots 1 \left| \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} \right| 0 \cdots$$

Let $\mathcal{W}_{q,n}$ be the set of such words of length n, $\mathcal{W}_q = \bigcup_{n \in \mathbb{N}} \mathcal{W}_{q,n}$.

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Ex.

11100101**0110**0001 is not 2-decreasing ($2 \cdot 1 \not> 2$)

01111 is not π-decreasing ($\pi \cdot 1 \not> 4$)

001111 is π-decreasing ($\pi \cdot 2 > 4$)

1-decreasing words, $\mathcal{W}_1$

In particular, in a 1-decreasing word every run of 0s is immediately followed by a strictly shorter run of 1s.

$$\cdots 1 \left| \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} \right| 0\cdots \qquad a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	3	5	8	Fibonacci

0	00	000	0000	…
1	10	001	0001	
	11	100	0010	
		110	1000	
		111	1001	
			1100	
			1110	
			1111	

2-decreasing words, $\mathcal{W}_2$

$$\cdots 1 \left| \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} \right| 0\cdots \qquad \text{where } 2a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	4	7	13	Tribonacci

0	00	000	0000	…
1	01	001	0001	
	10	010	0010	
	11	100	0011	
		101	0100	
		110	0101	
		111	1000	
			1001	
			1010	
			1100	
			1101	
			1110	
			1111	

A new paper in preparation with Sergey Dovgal...

- Fibonacci Cubes with Applications and Variations.
  Ömer Eğecioğlu, Sandi Klavžar and Michel Mollard
  World Scientific, 2023
- Q-bonacci words and numbers. Sk, Fibonacci conference
  https://kirgizov.link/talks/fiboconf.pdf
  The Fibonacci Quarterly, 2022, https://arxiv.org/abs/2201.00782
- Combinatorial Gray codes-an updated survey, Torsten Mütze
  https://arxiv.org/pdf/2202.01280.pdf
  to appear in Electronic Journal of Combinatorics
- Asymptotic bit frequency in Fibonacci words. BKV, GASCom 2022
  https://kirgizov.link/talks/gascom2022.pdf
  Pure Mathematics and Applications, 2022, https://arxiv.org/abs/2106.13550
- Gray codes for Fibonacci q-decreasing words.
  Jean-Luc Baril, Sk and Vincent Vajnovszki
  Theoretical Computer Science, 2022, https://arxiv.org/abs/2010.09505
- Fibonacci-run graphs I: Basic properties. Ömer Eğecioğlu and Vesna Iršič
  Discrete Applied Mathematics, 2021, https://arxiv.org/abs/2010.05518
- Qubonacci words. BKV
  Permutations patterns 2021, https://kirgizov.link/talks/qubonacci.pdf

From Sturmian prefixes starting,
Traversing decreasing words,
Discover a beautiful function,
United in fractal of sherds!

From Sturmian to q-decreasing

E.g., slope is $q = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}}$

Sturmian word $s(1/\varphi) = 0100101001001010...$ *(aka Fibonacci word)*

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1. Construct q-suffixes from sturmian prefixes ending with 1

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2. **Allow $\widehat{0}$ as q-suffix.**

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Length 2: $11, 1\widehat{0}, \widehat{0}\widehat{0}$
Length 3: $111, 11\widehat{0}, 1\widehat{0}\widehat{0}, \widehat{001}, \widehat{0}\widehat{0}\widehat{0}$

From Sturmian to q-decreasing

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Cards.: 1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, 286, 448, ...

From Sturmian to q-decreasing. Natural case

E.g., slope is $q = 2$

Sturmian word $s(2) = 101101101101101...$

From Sturmian to q-decreasing. Natural case

E.g., slope is $q = 2$

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$$\begin{aligned} {\color{red}1} &\to \widehat{01} \\ {\color{red}101} &\to \widehat{0011} \\ {\color{red}1011} &\to \widehat{00111} \\ &\ldots \end{aligned}$$

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...
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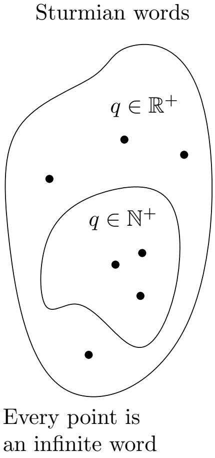
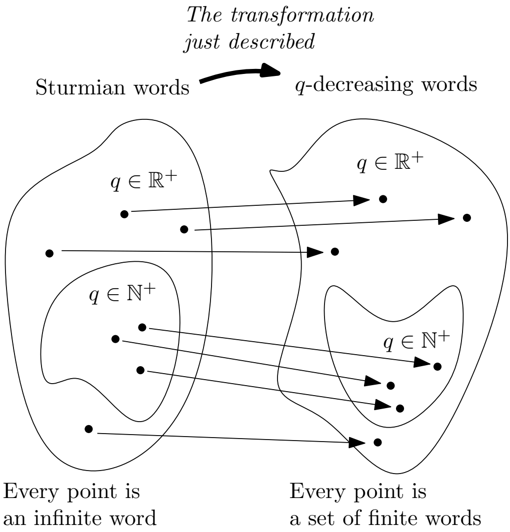
Cards.: 1, 2, 4, 7, 13, 24, ...

General picture



The picture is draw in collaboration with Sima and Fedia.

General picture



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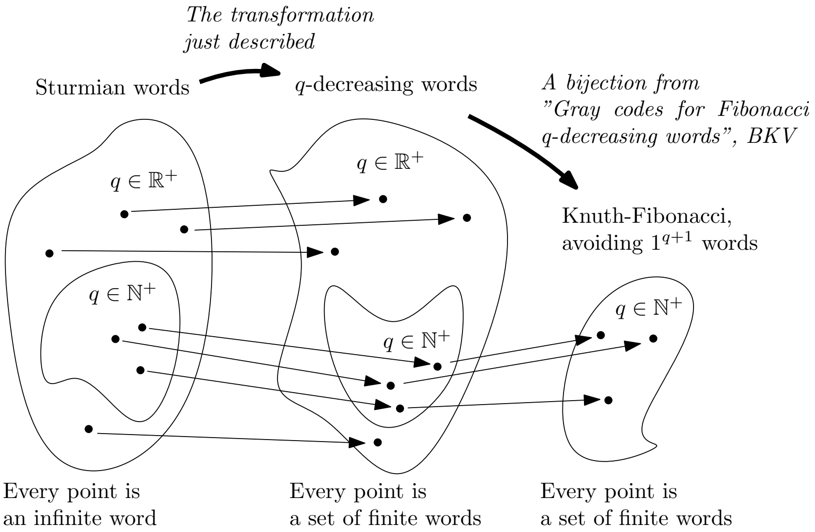

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Growth ratio

q-decreasing words. Growth ratio.

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$$\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|} \ ?$$

Consider the following function

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For $q = 2$, it is the tribonacci constant.

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For $q = 5/3$?

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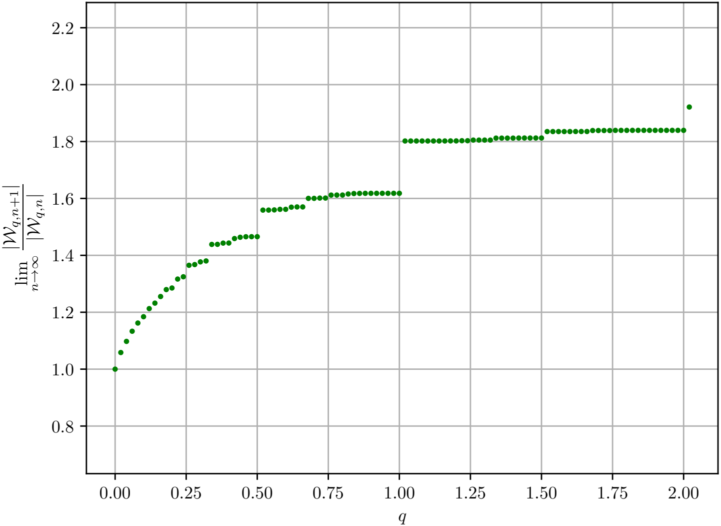
For $q = 2$, it is the tribonacci constant.
For $q = 5/3$?
For $q = \varphi$?

GeneℝAlization of the golden ratio



$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ as a function of q.

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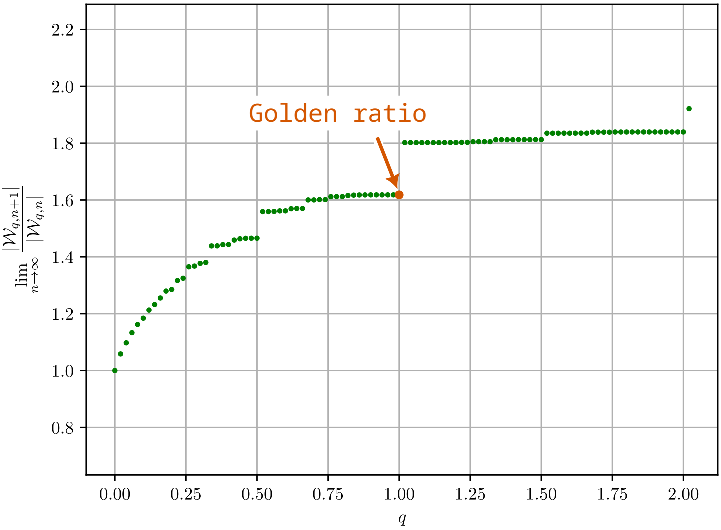

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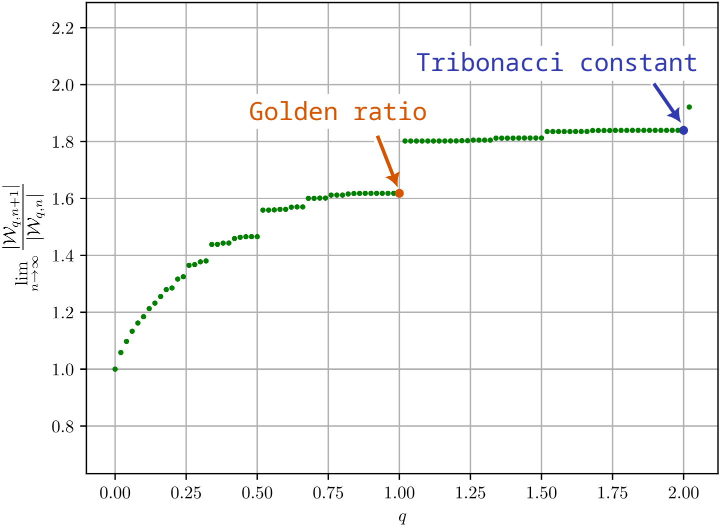

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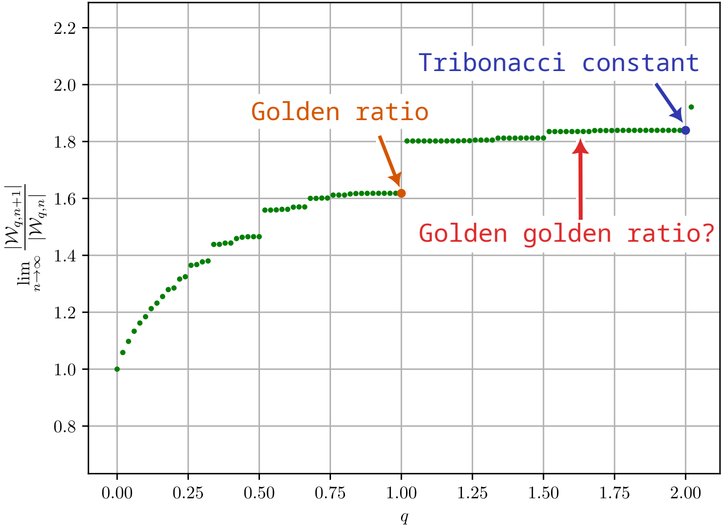

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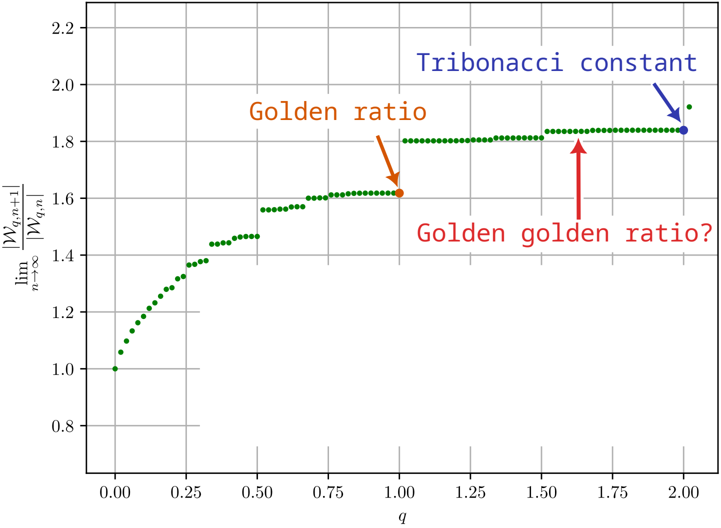

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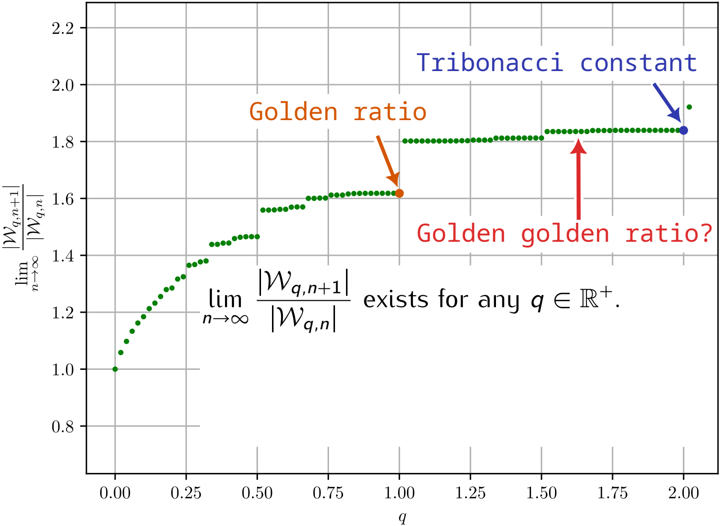
Gene$\mathbb{R}$alization of the golden ratio



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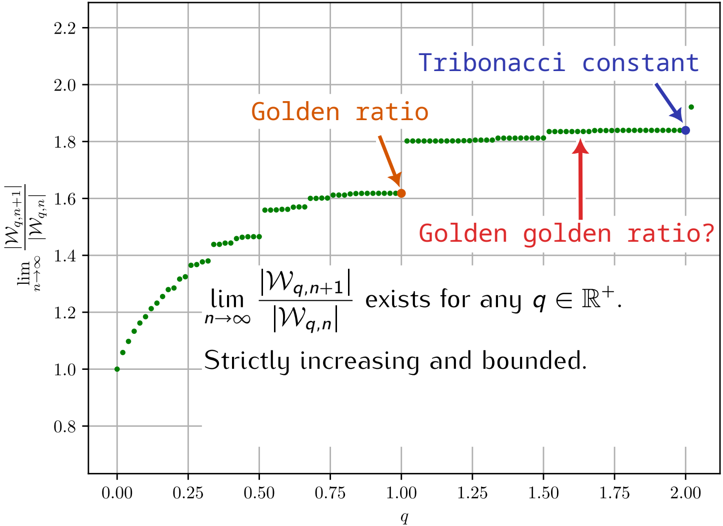

$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ exists for any $q \in \mathbb{R}^+$.

Strictly increasing and bounded.

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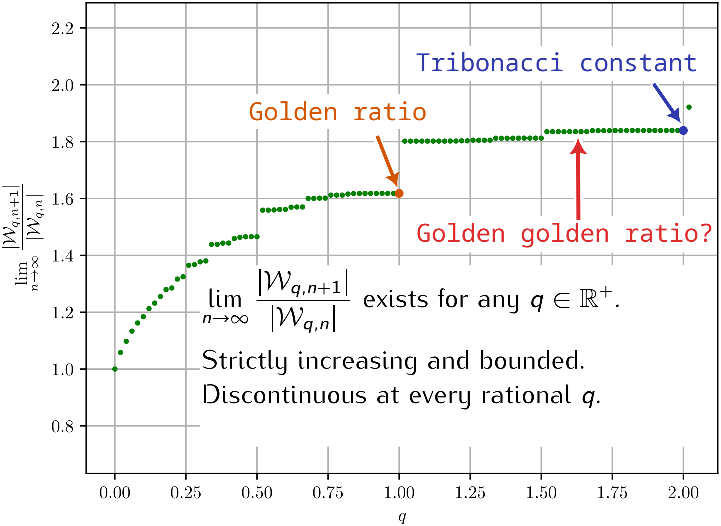

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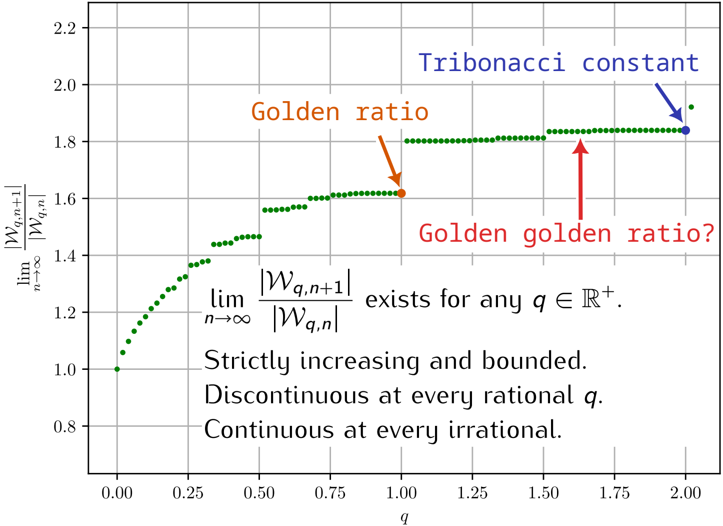


$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ exists for any $q \in \mathbb{R}^+$.

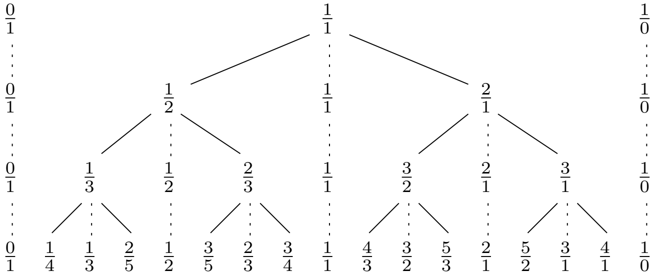
Strictly increasing and bounded.
Discontinuous at every rational q.
Continuous at every irrational.

$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ as a function of q.

Fractal

Stern, Brocot and Minkowski

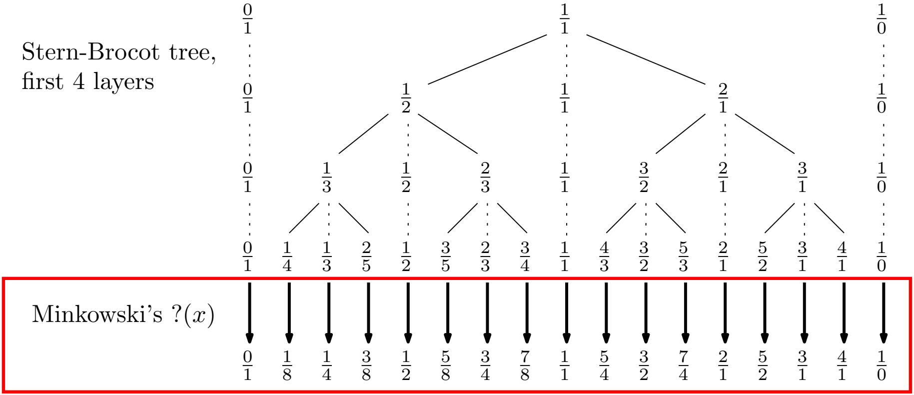
Stern-Brocot tree, first 4 layers



Mediant of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$, it is used to construct the tree.

It is also called a *freshman sum*.

Stern, Brocot and Minkowski



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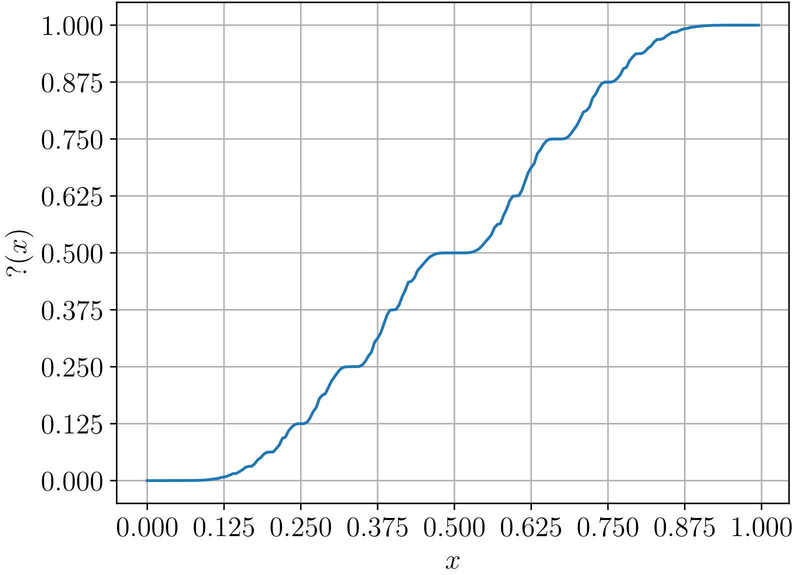
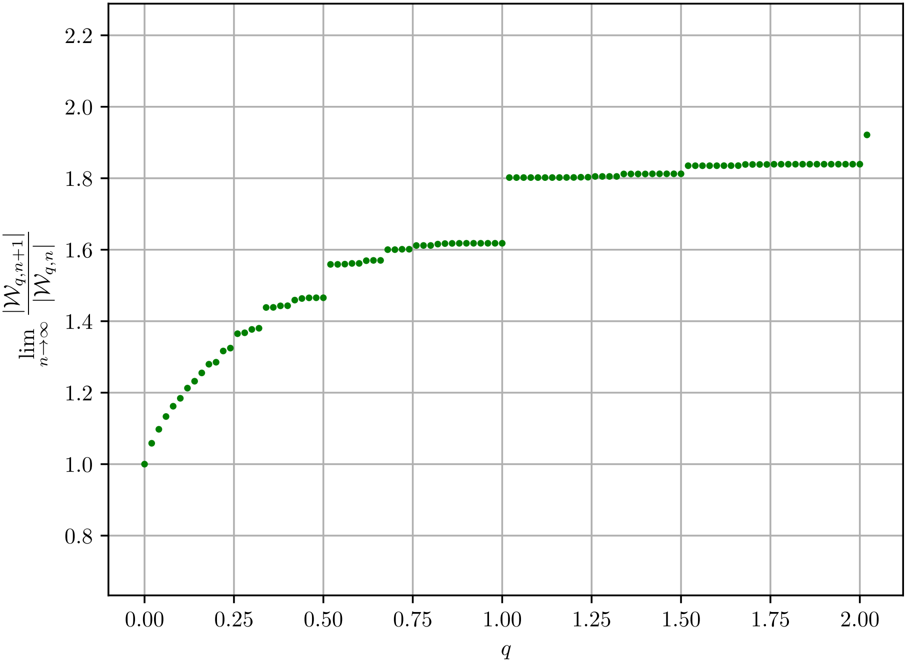
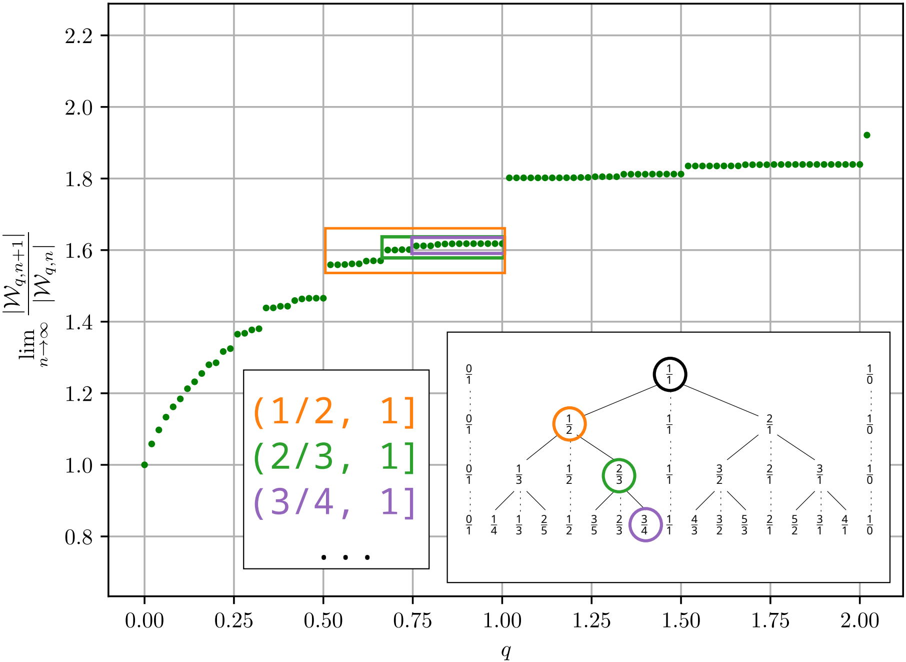
$?(x)$ **maps a rational to dyadic rational** $\frac{x}{2^y}$

Minkowski's question-mark function



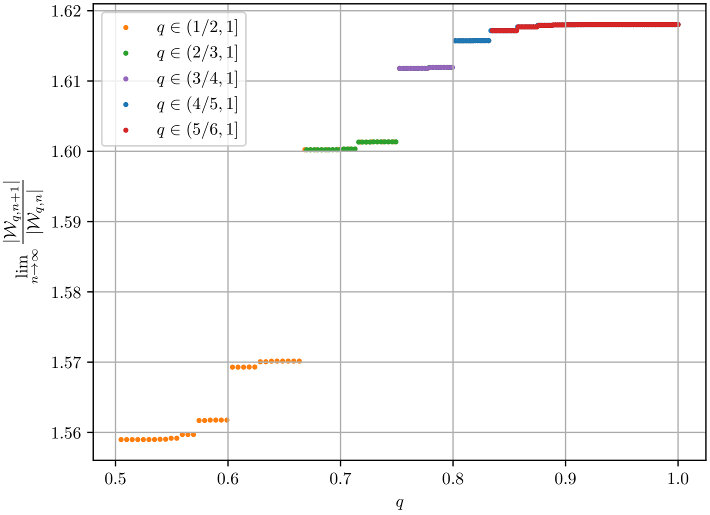
Then $?(x+1) = 1+?(x)$ for $x > 1$.

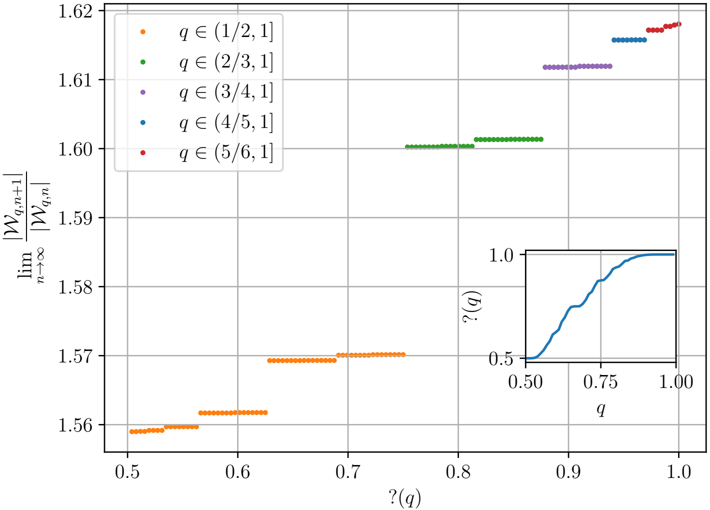

2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
q



$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$
q
(1/2, 1]
(2/3, 1]
(3/4, 1]
...

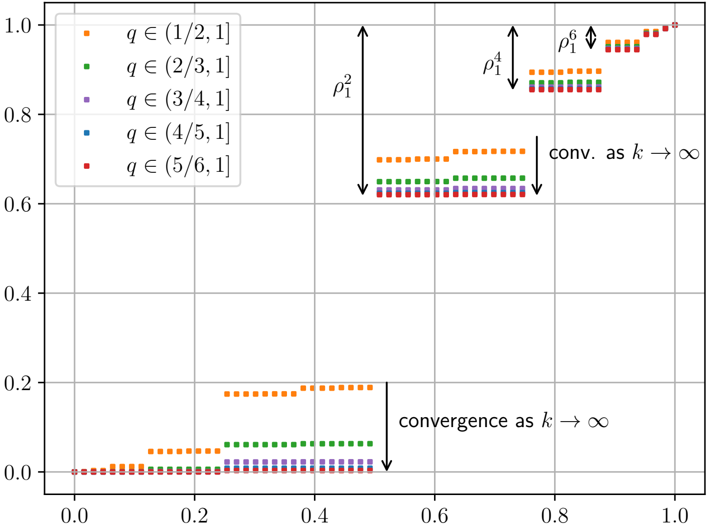
Intervals $(k/(k+1), 1]$ before rescaling




1.62
1.61
1.60
1.59
1.58
1.57
1.56
$q \in (1/2, 1]$
$q \in (2/3, 1]$
$q \in (3/4, 1]$
$q \in (4/5, 1]$
$q \in (5/6, 1]$
$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$
0.5
0.6
0.7
0.8
0.9
1.0
$?(q)$
1.0
0.5
0.50
0.75
1.00
q

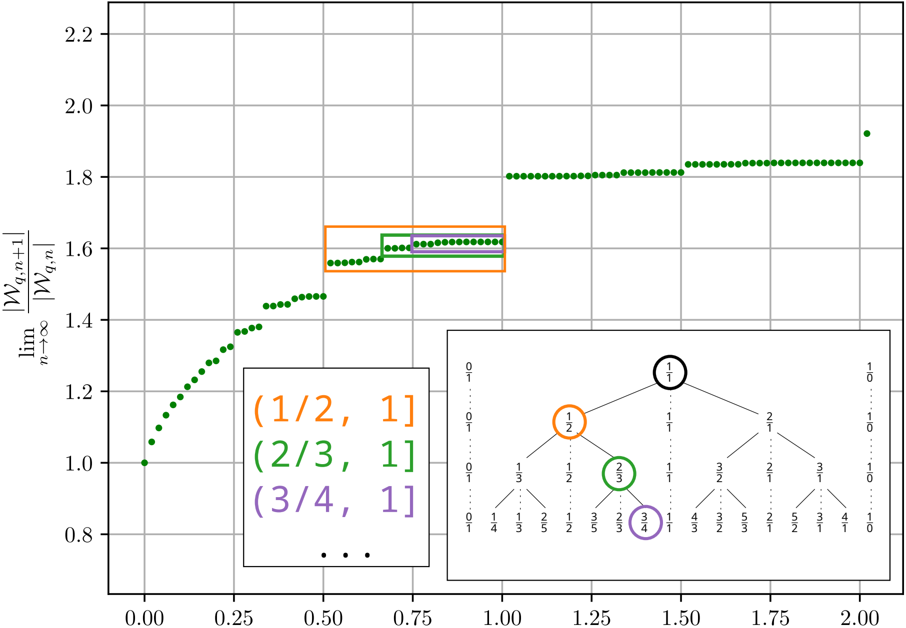
Map x to $?(x)$.

$(k/(k+1), 1]$, regions rescaled and superimposed

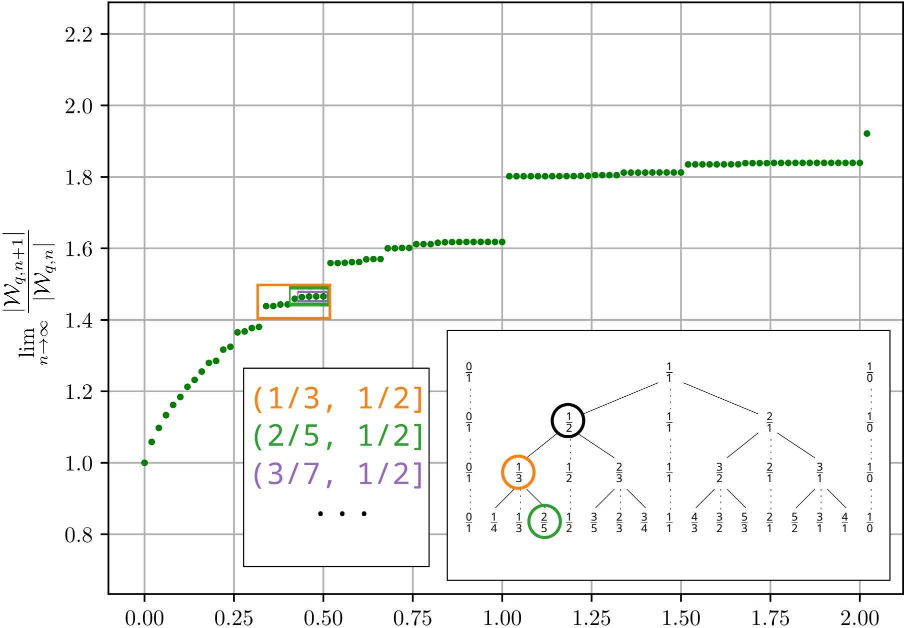


Rescale y to be inside $[0, 1]$.
Map x to $?(x)$ and rescale the result to be inside $[0, 1]$

Different families of regions have different limits

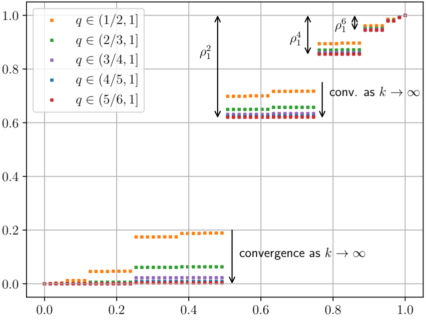


Different families of regions have different limits

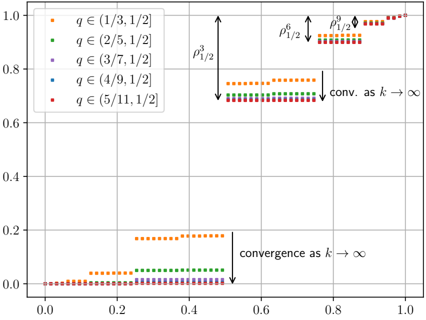


Different families of regions rescaled and superimposed

$(k/(k+1), 1]$



$(k/(1+2k), 1/2]$



Where $\rho_{\frac{c}{d}}$ is the root of $1 - x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\lfloor \frac{id}{c} \rfloor}$.

E.g. $\rho_1 = \dfrac{1}{\varphi} = \dfrac{2}{1+\sqrt{5}}$ is the root of $1 - x^2 - x$.

Any proofs ?

G.f. $W_q(x) = \dfrac{1}{(1-x)\left(1-\sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}\right)}.$

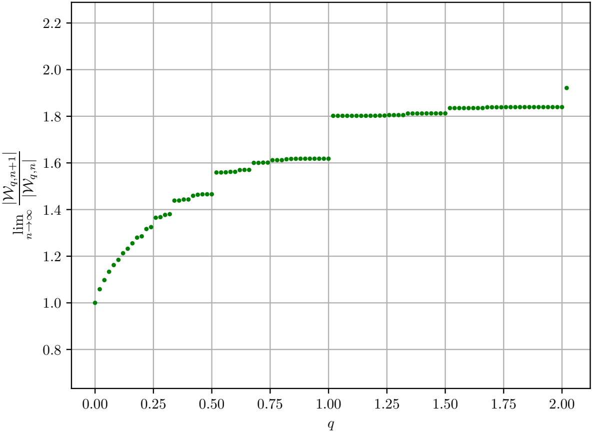
If $q = \dfrac{c}{d} \in \mathbb{Q}^+$ the g.f. is

$$W_{\frac{c}{d}}(x) = \frac{1-x^{c+d}}{(1-x)\left(1-x^{c+d}-\sum_{i=0}^{c-1} x^{1+i+\left\lfloor \frac{id}{c} \right\rfloor}\right)}.$$

We represent polynomial denominators of generating functions as certains subsets of points in $\mathbb{Z}^2$...

We use Pick's Theorem and certain algebraico-analytico-combinatorial gymnastics to prove the results.

Open question: which jumps are higher?



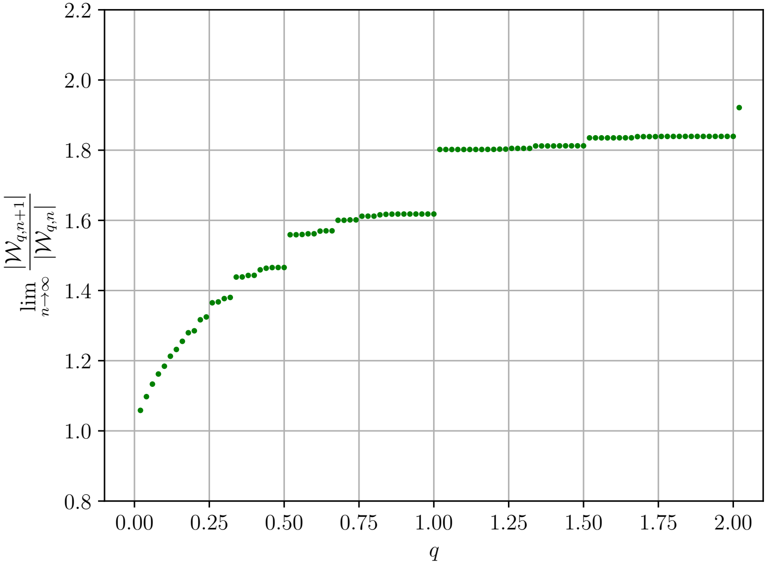
Here is the sequence of positive rational numbers ordered by corresponding jumps of the function $\Phi(q) = \lim_{n\to\infty} |\mathcal{W}_{q,n+1}|/|\mathcal{W}_{q,n}|$

$$1, \frac{1}{2}, 2, \frac{1}{3}, \frac{1}{4}, 3, \frac{2}{3}, \frac{1}{5}, \frac{1}{6}, \frac{3}{2}, \frac{1}{7}, 4, \frac{2}{5}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{3}{4}, \frac{1}{11}, \frac{2}{7}, \frac{1}{12}, 5, \frac{3}{5}, \frac{1}{13}, \frac{4}{3}, \frac{1}{14}, \frac{2}{9}, \frac{1}{15}, \frac{1}{16}, ?$$

Next term is ... ???

ArXiv preprint is coming soon!
We thank you so much
For staying in tune.

Minkowski's scaling



Minkowski's scaling