Structure and growth of $\mathbb{R}$-bonacci words

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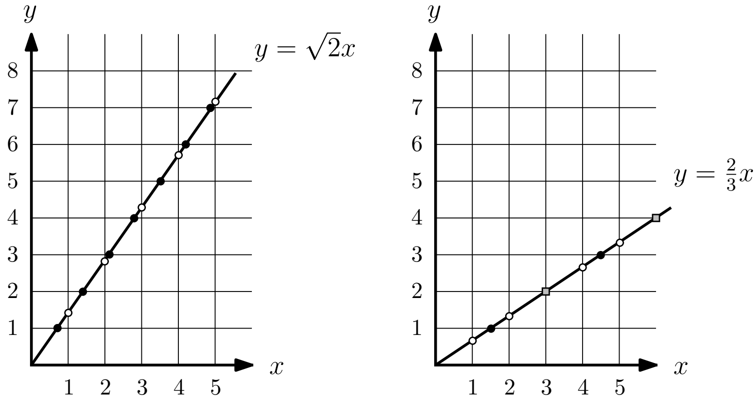
Abstract

A binary word is called q-decreasing, for $q > 0$, if every of its length maximal factors of the form 0^a1^b, $a > 0$, satisfies $q \cdot a > b$. We bijectively link q-decreasing words with certain prefixes of the cutting sequence of the line $y = qx$. We show that the number of q-decreasing words of length n grows as $\Phi(q)^n C_q$ for some constant C_q which depends on q but not on n. We demonstrate that $\Phi(1)$ is the golden ratio, $\Phi(2)$ is equal to the tribonacci constant, $\Phi(k)$ is $(k+1)$-bonacci constant. Furthermore, we prove that the function $\Phi(q)$ is strictly increasing, discontinuous at every positive rational point, exhibits a fractal structure related to the Stern–Brocot tree and Minkowski's question mark function.

1 Introduction

For any real $q > 0$, the *ray cutting word* $s(q)$ is defined as an intersection sequence of a straight half-line $y = qx$ for $x \in (0, \infty)$ with the lines of a square grid ($y = i$ or $x = i$ for $i \in \mathbb{N}^+$). Going along the half-line, starting from $(0, 0)$, we write `1` if the line intersects a horizontal edge and `0` in case of a vertical edge (see Figure 1), we write `01` (in this order) when crossing an intersection point of grid lines.

For any irrational slope q, the word $s(q)$ is aperiodic and Sturmian. In general setting, Sturmian words are defined as cutting sequences of the line $y = ax + b$ for $x \in (0, \infty)$, irrational $a > 0$ and real $b \in [0, 1)$ or equivalently as binary words having exactly $n + 1$ factors (contiguous subwords) of length n. Sturmian words shine in several different areas of mathematics: combinatorics, number theory, tilings, discrete dynamical systems. The structures similar to Sturmian words were already studied by Johann III Bernoulli [4] in 1771. The exposition of Sturmian words and related results can be found in Chapter 2 (written by Berestel and Séébold) of Lothaire's œuvre [15] and in the book of Allouche and Shallit [1]. For rational slope q, the word $s(q)$ is periodic, its smallest factor f such that $s(q) = f \cdot f \cdot f \ldots$, where $\cdot$ means concatenation, corresponds to the Christoffel word of slope q [5, 7].



$s(\sqrt{2}) = 101011010110\ldots$ $\qquad$ $s\left(\frac{2}{3}\right) = 0100101001\ldots$

Figure 1: Cutting sequences with slopes $\sqrt{2}$ and $\frac{2}{3}$.

One paradigmatic example of Sturmian words is the *Fibonacci word* `010010100100101...` which is characterized by a cutting sequence of the line with a slope $1/\varphi$, where $\varphi = (1+\sqrt{5})/\sqrt{2}$. It can also be obtained either by a recursive simultaneous application of substitution rules $\{\texttt{0} \mapsto \texttt{01}, \texttt{1} \mapsto \texttt{0}\}$ to an initial string `0`, or as a limit of recursive concatenations of strings $S_n = S_{n-1}S_{n-2}$, where $S_0 = \texttt{0}$ and $S_1 = \texttt{01}$.

Now consider another Fibonacci object, (or, more generally, k-bonacci), which is an ensemble of binary words of length n avoiding k consecutive `1`s. It seems that this object appears for the first time in Knuth's book [14, p. 286]. The set of such words is in bijection with tilings of stripes of length $(n+1) \times 1$ with tiles of size 1×1 (monomers), 2×1 (dimers), $\ldots$, $k \times 1$ (k-mers), so it is convenient to call them *k-bonacci tilings*. The cardinality of the set of such words of length n is equal to nth k-bonacci number (see Feinberg [10] and Miles [16]), which is obtained by a recurrence relation $a_n = a_{n-1} + a_{n-2} + \ldots + a_{n-k}$ with initial conditions $a_0 = a_{-1} = 1$ and $a_j = 0$ for any $j < -1$.

These two Fibonacci objects belong to two seemingly different worlds. In this paper we propose a link between these worlds: we show that certain subsets of prefixes of ray cutting words can be used as building blocks to construct generalized Fibonacci tilings. To demonstrate this link, we extend the family of q-decreasing words, defined in [3], to cover all positive real numbers as possible values of the parameter q.

Definition 1. For $q \in \mathbb{R}^+$, a binary word is called q-decreasing, if every of its length maximal factors of the form $\texttt{0}^a\texttt{1}^b$, $a > 0$, satisfies $q \cdot a > b$.

We denote by $\mathcal{W}_{q,n}$ the set of *q-decreasing words* of length n, Table 1 gives some examples. It is interesting to note that Eğecioğlu and Iršič [9] independently discovered and studied hypercube subgraphs associated with a subset of words from $\mathcal{W}_{1,n}$. By giving a Gray code for $\mathcal{W}_{1,n}$, Baril, Kirgizov and Vajnovszki [3] prove the Eğecioğlu-Iršič conjecture [9] about the existence of a Hamiltonian path in such hypercube subgraphs. In the same paper [3] it has been shown that q-decreasing words are in bijection with k-bonacci tilings, where $k = q + 1$, i.e. with the set of n-length binary words that avoid

n	1	2	3	4	5
$\mathcal{W}_n^{\sqrt{2}}$	0 1	00 01 10 11	000 001 010 100 101 110 111	0000 0001 0010 0011 0100 0101 1000 1001 1010 1100 1101 1110 1111	00000 00001 10010 00010 10011 00011 10100 00100 10101 00101 11000 00110 11001 01000 11010 01001 11100 01010 11101 10000 11110 10001 11111
$\lvert\mathcal{W}_n^{\sqrt{2}}\rvert$	2	4	7	13	23

n	1	2	3	4	5
$\mathcal{W}_n^{2/3}$	0 1	00 10 11	000 001 100 110 111	0000 0001 0010 1000 1001 1100 1110 1111	00000 00001 00010 00100 10000 10001 10010 11000 11001 11100 11110 11111
$\lvert\mathcal{W}_n^{2/3}\rvert$	2	3	5	8	12

Table 1: q-decreasing words for $n \in [1,5]$ and $q \in \{\sqrt{2}, \frac{2}{3}\}$

$q+1$ consecutive 1s. In Section 2 we decompose q-decreasing words into sequences of words corresponding to ray cutting prefixes ending on 1.

The number $\Phi(q)$, called the exponential growth constant and is defined as the limit ratio of successive cardinalities $\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$. In Sections 3 and 4 we show that this limit exists, and explore the structure of $\Phi(q)$ as a function of q. It turns out that the function $\Phi(\cdot)$ is bounded, discontinuous at every positive rational point, is strictly increasing over $(0,\infty)$, and also exhibits a nice fractal structure (see Figure 2), which is reminiscent of fractal structures in other information-theoretical applications, such as [2]. We show that at the vicinity of each rational point q, $\Phi(q)$ converges locally to a piecewise linear function.

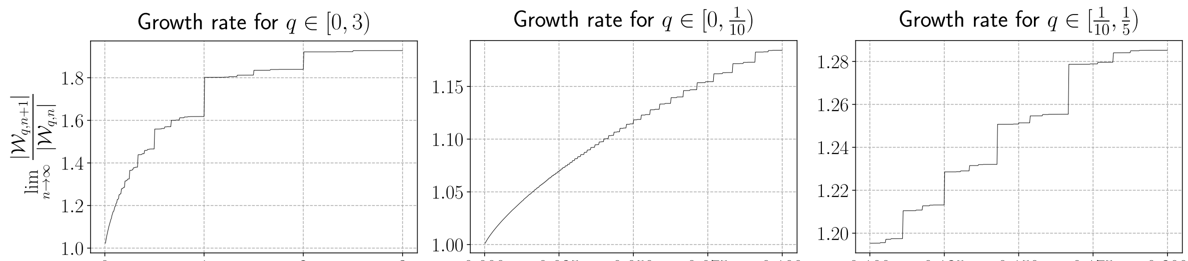


Figure 2: $\lim_{n\to\infty} |\mathcal{W}_{q,n+1}|/|\mathcal{W}_{q,n}|$ as a function of q in three different intervals. This function is jump discontinuous at every positive rational point.

2 Contruction from ray cutting prefixes

Here we express q-decreasing words as sequences of ray cutting prefixes ending on 1. It is handy to use Kleene star operator (it corresponds to `SEQ` operator in Flajolet–Sedgewick book [11]), which constructs a disjoint union of finite concatenations from strings of a given family. For instance, $(\{\mathtt{0},\mathtt{10}\})^*$ provides all binary strings which are empty or end on $\mathtt{0}$ and do not contain two consecutive $\mathtt{1}$s. We also use the "$\cdot$" symbol to denote all possible pairwise concatenations between the elements of two families.

Proposition 1. *For $q \in \mathbb{R}^+$, the set $\mathcal{W}_q$ of q-decreasing words can be represented as*

$$\mathcal{W}_q = (\{\mathtt{1}\})^* \cdot (\mathcal{S}_q)^*, \ \ \textit{where} \ \ \mathcal{S}_q = \cup_{i=0}^{\infty}\{\mathtt{0}^{1+\lfloor i/q \rfloor}\mathtt{1}^i\}.$$

Proof. By definition 1, a q-decreasing word is a concatenation of factors $\mathtt{0}^a\mathtt{1}^b$ satisfying $a = 0$ or $qa > b$. If $a = 0$, the string starts with an arbitrary sequence of $\mathtt{1}$s, which is $(\{\mathtt{1}\})^*$, otherwise the condition $qa > b$ can be rewritten as $a \geqslant 1 + \lfloor b/q \rfloor$. By grouping the extra zeros at the beginning of each factor $\mathtt{0}^a\mathtt{1}^b$, we write it as $\mathtt{0}^t\mathtt{0}^{1+\lfloor i/q \rfloor}\mathtt{1}^i$ with $i = b$ and $t = b - a$. Furthermore, since $\mathtt{0} \in \mathcal{S}_q$, the factor $\mathtt{0}^t$ belongs to the family $(\mathcal{S}_q)^*$, and the remaining part $\mathtt{0}^{1+\lfloor i/q \rfloor}\mathtt{1}^i$ belongs to $\mathcal{S}_q$ itself. This procedure allows us to decompose the remainder into a sequence of strings from $\mathcal{S}_q$, which finishes the proof. □

For a binary word α containing n $\mathtt{0}$s and m $\mathtt{1}$s, we define a transformation $\kappa(\alpha) = \mathtt{0}^{n+1}\mathtt{1}^m$, so that the empty word ϵ is mapped to the word $\mathtt{0}$. We provide a decomposition of q-decreasing words into partitions of certain ray cutting prefixes by using the above transformation.

Proposition 2. *For $q \in \mathbb{R}^+$, the transformation κ bijectively maps the set of prefixes ending with 1 of the ray cutting word $s(q)$ to the set $\mathcal{S}_q$, which is used in the construction of q-decreasing words.*

Proof. Take a ray cutting word $s(q) = s_1s_2s_3s_4\ldots$ where every s_i is a binary digit. The index of ith $\mathtt{1}$ in this word is $i + \left\lfloor \frac{i}{q} \right\rfloor$. The prefix $s_1s_2\ldots s_{i+\lfloor i/q \rfloor}$ of $s(q)$ contains exactly i $\mathtt{1}$s and $\left\lfloor \frac{i}{q} \right\rfloor$ $\mathtt{0}$s. The word $\kappa(s_1s_2\ldots s_{i+\lfloor i/q \rfloor}) = \mathtt{0}\mathtt{0}^{\left\lfloor \frac{i}{q} \right\rfloor}\mathtt{1}^i$ is a factor from the set $\mathcal{S}_q$, which completes the proof. □

q	Ray cutting word	Factors from $\mathcal{S}_q$	Some q-decreasing words
$\sqrt{2}$	101011010110...	$\kappa(\epsilon) = 0$, $\kappa(1) = 01$, $\kappa(101) = 0011$, $\kappa(10101) = 000111$, ...	111100011000111, 000111101000001, 100000101000001
$\frac{2}{3}$	010010100101...	$\kappa(\epsilon) = 0$, $\kappa(01) = 001$, $\kappa(01001) = 000011$, $\kappa(0100101) = 00000111$, ...	111100001100001, 000011001000001, 001000000001111

Table 2: Illustration of the transformation κ. Prefixes ending with $\mathtt{1}$ of the ray cutting word $s(q)$ correspond to factors from the set $\mathcal{S}_q$.

q	Ray cutting word	Counting Sequence $\lvert\mathcal{W}_{q,n}\rvert$	OEIS
$\frac{1}{2}$	0010010010010010...	1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, ...	Narayana's cows, A930
$1/\varphi$	0100101001001010... Fibonacci word	1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, 286, 448, ...	NEW
$\frac{2}{3}$	0100101001010010...	1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, 286, 449, ...	Comp. into 1s, 3s and 5s, A60961
1	0101010101010101...	1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...	Fibonacci, A45
2	1011011011011011...	1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...	Tribonacci, A73
$\frac{3}{2}$	1010110101101011...	1, 2, 4, 7, 13, 23, 42, 76, 138, 250, 453, 821, 1488, 2697, ...	NEW
$\sqrt{2}$	1010110101101010...	1, 2, 4, 7, 13, 23, 42, 76, 138, 250, 453, 821, 1488, 2697, ...	NEW
φ	1011010110110101...	1, 2, 4, 7, 13, 24, 44, 81, 148, 272, 499, 916, 1681, 3085, ...	NEW
e	1101110111011011...	1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...	NEW
π	1110111011101110...	1, 2, 4, 8, 16, 31, 61, 120, 236, 463, 910, 1788, 3513, 6901, ...	NEW

Table 3: Examples of ray cutting words and corresponding counting sequences for the cardinalities of q-decreasing words.

3 Rational discontinuity

In this section we study the function $\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$, see Figure 2. We know (see e.g. [11]) that if $P(x)/Q(x)$ is a rational function and α is the unique smallest in modulus root of multiplicity 1 of polynomial $Q(x)$, then the coefficient at x^n of the generating function $P(x)/Q(x)$ grows asymptotically as $C\alpha^{-n}$ as $n \to \infty$, for some constant C.

Note that Proposition 1 yields the generating function $W_q(x) = \sum_{n=0}^{\infty} |\mathcal{W}_{q,n}|x^n$ of the family $\mathcal{S}_q$ for any $q \in \mathbb{R}^+$:

$$W_q(x) = \frac{1}{(1-x)\left(1 - \sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}\right)}. \tag{1}$$

The case where q is a positive *rational number* represented by an irreducible fraction $\frac{c}{d}$ is treated in [13] where the author expresses the generating function $W_q(x)$ as

$$W_{q=\frac{c}{d}}(x) = \frac{1 - x^{c+d}}{(1-x)\left(1 - x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}\right)}. \tag{2}$$

Furthermore, Equation (1) holds for any positive real q, and is more general, although simpler, form of Equation (2) which is only valid for $q \in \mathbb{Q}^+$.

Assume that $q \in \mathbb{Q}^+$ is represented by an irreducible fraction c/d. From Equation (2) we see that the growth rate is dictated by the smallest in modulus root, denoted by ρ_q, of the polynomial

$$\Pi_q := 1 - x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}.$$

Lemma 1. *The smallest in modulus root ρ_q of Π_q is positive and real.*

Proof. Let R be the positive real value for which monotonically increasing on $(0,\infty)$ function $f(x) = x^{c+d} + \sum_{i=0}^{c-1} x^{1+i+\lfloor \frac{i}{q} \rfloor}$ achieves value $f(R) = 1$. For any z such that $|z| < R$, we have, by triangle inequality, $|f(z)| \leqslant f(|z|) < f(R) = 1$. Since $f(x)$ is strongly aperiodic (there does not exist a function h analytic at 0 and an integer $d \geqslant 2$ such that $f(z) = h(z^d)$, see Daffodil Lemma and related discussions on pp. 293–295 from Flajolet–Sedgewick book [11]), for all complex points $z \neq R$ such that $|z| = R$ we also have $|f(z)| \neq 1$. □

We have $\Phi(q) = 1/\rho_q$, and also, since words from $\mathcal{W}_{q,n}$ form a subset of all binary words, we have $1/2 \leqslant \rho_q \leqslant 1$. Comparing Equations (1) and (2) we see that Π_q shares the same smallest in modulus root with

$$1 - \sum_{i=0}^{\infty} x^{1+i+\lfloor \frac{i}{q} \rfloor}.$$

It also shares the smallest in modulus root with the power series

$$A_q := 1 - \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} x^{1+i+\lfloor \frac{i}{q} \rfloor + j},$$

the fact that has a nice geometrical interpretation. To see it, we have to decompose the set $\mathcal{W}_q$ in another way, different from what was given in Section 2. Here we use a set $\mathcal{F}_q$ of factors $0^a 1^b$ such that $qa > b$ and $b \geqslant 1$. With this, we decompose any word $w \in \mathcal{W}_q$ as a sequence of 1s, followed by a sequence of factors from $\mathcal{F}_q$, followed by a sequence of 0s. Any of these sequences can be empty. We have

$$w = \overbrace{\texttt{1...1}}^{\text{some ones}} \; \overbrace{f_1 f_2 \ldots . f_k}^{f_\ell \in \mathcal{F}_q} \; \overbrace{\texttt{0...0}}^{\text{some zeros}}, \quad \text{where } \mathcal{F}_q = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{\infty} \{ \; \overbrace{\texttt{0...00}}^{1+\lfloor \frac{i}{q} \rfloor + j \text{ zeros}} \underbrace{\texttt{1...11}}_{i \text{ ones}} \; \}.$$

Now, we write the g.f. $W_q(x)$ as

$$W_q(x) = \frac{1}{1-x} \cdot \frac{1}{A_q} \cdot \frac{1}{1-x}.$$

Consider the grid $\mathbb{Z}^+ \times \mathbb{Z}^+$, and make every point (a, b) correspond to a factor $\texttt{0}^a \texttt{1}^b$. The power series A_q sums over all points with positive integer coordinates found under the line $b = qa$. Figure 3 gives some examples.

Proposition 3. *The function* $\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ *is*

- **a)** *strictly increasing over* $q \in [0, \infty)$*;*
- **b)** *bounded,* $1 \leqslant \Phi(q) < 2$*, with* $\Phi(0) = 1$ *and* $\lim_{q\to\infty} \Phi(q) = 2$*;*
- **c)** *left-continuous (and right-discontinuous) at every positive rational point;*
- **d)** *continuous at every positive irrational point.*

Proof. For any $x \in [0, 1)$, using the classical result about geometric series $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$, we obtain that

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} x^{1+i+\lfloor \frac{i}{q} \rfloor + j} \leqslant \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x^{i+j} \leqslant \sum_{i=2}^{\infty} x^i \cdot \sum_{j=0}^{\infty} x^j \leqslant \frac{x^2}{(1-x)^2}.$$

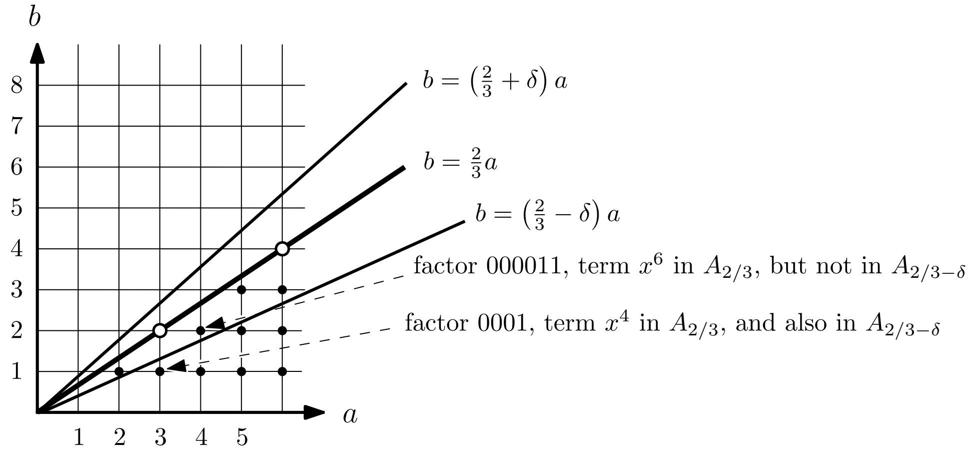


Figure 3: The line $b = \frac{2}{3}a$ and the geometrical interpretation of factors 0^a1^b where $\frac{2}{3}a > b$.

So, the series A_q evaluated at $x \in [0, 1)$ is bounded, thus convergent for any real $q \in (0, \infty)$. Also recall that the zero of A_q, denoted by ρ_q, is located between 1/2 and 1.

Let $\delta > 0$.

a) Note that $A_{q+\delta} = 1 - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q+\delta}\right\rfloor+j}$ contains all terms of $A_q = 1 - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q}\right\rfloor+j}$ together with terms not presented in A_q. This can be seen geometrically on Figure 3. So, we necessarily have $\rho_{q+\delta} < \rho_q$, and thus $\Phi(q) < \Phi(q+\delta)$.

b) If $q = 0$, the only q-decreasing word of length n is 1^n, so $\Phi(0) = 1$. It is not difficult to see that $\lim_{q\to 0}\Phi(q) = 1$. As $q \to \infty$ we allow more and more binary words, and the functional limit of A_q can be expressed as 1 minus the sum over all integer points from the positive quadrant: $\lim_{q\to\infty} A_q(x) = 1 - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x^{i+j}$, so $\lim_{q\to\infty}\Phi(q) = 2$.

c) For a positive rational q represented by an irreducible fraction $\frac{c}{d}$ the line $b = qa$ contains points of integer coordinates (kd, kc) for $k \in [1, \infty)$. For any $\delta > 0$, these points are below the line $b = (q+\delta)a$. The corresponding terms of the form x^{kd+kc} are included in $A_{q+\delta}$ but not in A_q. This, in turn, influences the smallest in modulus root of $A_{q+\delta}$ which is strictly less than the smallest in modulus root of A_q and we obtain $\Phi(q) < \lim_{\delta\to 0^+}\Phi(q+\delta)$. The difference between two limits can be explicitly calculated (see Proposition 4). In other case, no line of the form $b = (q-\delta)a$ can have the points (kd, kc) below it (even if $\delta = 0$). Any point lying under $b = qa$ also lies under $b = (q-\delta)a$ for sufficiently small δ. We obtain $\lim_{\delta\to 0^+}\Phi(q-\delta) = \Phi(q)$.

d) For a positive irrational q, the line $b = qa$ contain no points of positive integer coordinates. So, the corresponding A_q can be approached by A_r where $r \to q$ from the right or from the left. □

Proposition 4. *For any irreducible $q = \frac{c}{d} \in \mathbb{Q}^+$, $\lim_{\delta\to 0^+}\Phi(q+\delta)$ equals the reciprocal of the smallest in modulus root, denoted by ρ_q^+, of the polynomial*

$$\Pi_q^+ := 1 - (2-x)x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\left\lfloor \frac{i}{q}\right\rfloor}.$$

We have

$$\lim_{\delta\to 0^+}\Phi(q+\delta) - \Phi(q) = \frac{1}{\rho_q^+} - \frac{1}{\rho_q},$$

where ρ_q *is the smallest in modulus root of* $\Pi_q := 1 - x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\lfloor \frac{i}{q} \rfloor}$.

Proof. Assume that q is represented by an irreducible fraction c/d. We consider a set of binary words $\mathcal{W}_{q,n}^+$ whose every length maximal factor of the form $\mathtt{0}^a\mathtt{1}^b$ respects $aq \geqslant b$. (It differs from the definition of q-decreasing words which encloses a strict inequality). It is clear (see e.g. Figure 3) that the set $\mathcal{W}_{q+\delta,n}$ approaches $\mathcal{W}_{q,n}^+$ as $\delta \to 0$, factors $\mathtt{0}^{kd}\mathtt{1}^{kc}$ corresponding to the points of the form (kd, kc) are included in both sets, no points located strictly above the line $b = qa$ are considered in both cases. The set $\mathcal{W}_q^+$ is constructed as $\mathcal{W}_q^+ = (\{\mathtt{1}\})^* \cdot (\mathcal{S}_q^+)^*$, where $\mathcal{S}_q^+ = \{\mathtt{0}\} \cup \bigcup_{i=1}^{\infty} \{\mathtt{0}^{\lceil i/q \rceil}\mathtt{1}^i\}$ (c.f. Proposition 1).

Note that $\mathcal{S}_q^+ = \{\mathtt{0}\} \cup \mathcal{B}$, where $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$, and $\mathcal{B}_0 = \bigcup_{i=1}^{c-1} \{\mathtt{0}^{1+\lfloor i/q \rfloor}\mathtt{1}^i\} \cup \{\mathtt{0}^d\mathtt{1}^c\}$, and $\mathcal{B}_{j+1}$ is constructed by inserting the factor $\mathtt{0}^d\mathtt{1}^c$ after the last $\mathtt{0}$ in words from $\mathcal{B}_j$. So, the g.f. $W_q^+(x)$ of the words in $\mathcal{W}_q^+$ is

$$W_{q=\frac{c}{d}}^+(x) = \frac{1}{(1-x)\left(1 - \left(x + \frac{\sum_{i=1}^{c-1} x^{1+i+\lfloor \frac{i}{q} \rfloor} + x^{c+d}}{1 - x^{c+d}}\right)\right)} =$$

$$= \frac{1 - x^{c+d}}{(1-x)\left(1 + x^{c+d+1} - 2x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\lfloor \frac{i}{q} \rfloor}\right)} = \frac{1 - x^{c+d}}{(1-x) \cdot \Pi_q^+}.$$

Since $g(x) := x + \frac{\sum_{i=1}^{c-1} x^{1+i+\lfloor \frac{i}{q} \rfloor} + x^{c+d}}{1 - x^{c+d}}$ is strongly aperiodic (there does not exist a function h analytic at 0 and an integer $d \geqslant 2$ such that $g(z) = h(z^d)$), from Daffodil Lemma [11] it follows that there are no complex roots smaller than or equal in modulus to ρ_q^+. The claimed result is obtained by comparing this formula with Equation (2). □

4 The fractal

As we can see on Figure 2 the graph of the function $\Phi(q)$ shows a certain amount of self-similarity. We explain some aspects of this fractality in the following propositions. Firstly, we zoom into the intervals $q \in (\frac{k}{k+1}, 1]$ for $k \in [1, \infty)$, and then look into a more general setting. Let us recall that the smallest in modulus root of $1 - x - x^2$ is $\rho_1 = 1/\varphi$, where $\varphi = (1 + \sqrt{5})/2$. The polynomials Π_q and Π_q^+ are defined in Proposition 4.

Proposition 5. *For natural* $k \geqslant 1$*, the smallest in modulus root* $\rho_{k/(k+1)}$ *of* $\Pi_{k/(k+1)}$ *is*

$$\rho_1 + C\rho_1^{2k}(1 + o(1)), \quad \text{as} \quad k \to \infty,$$

where $C = \rho_1^3/(1 + 2\rho_1)$*, and* ρ_1 *is the smallest in modulus root of* $1 - x - x^2$.

Proof. After arithmetic transformations, the equation $\Pi_{k/(k+1)} = 0$, i.e. the equation

$$1 - x^{2k+1} - \sum_{i=0}^{k-1} x^{1+i+\lfloor \frac{i(k+1)}{k} \rfloor} = 0$$

turns into

$$1 - x^{2k+1} - \sum_{i=0}^{k-1} x^{1+2i} = 0 \quad \Leftrightarrow \quad 1 = x \cdot \frac{x^{2k+2} - 1}{x^2 - 1} \quad \Rightarrow \quad 1 = x + x^2 - x^{2k+3}.$$

We multiplied by (x^2-1) both sides of equation, adding two new roots 1 and -1. This does not change the overall picture of the asymptotics, because $0 < \rho_{k/(k+1)} < 1$. By Lemma 1 there is no complex roots smaller than of equal in modulus to $\rho_{k/(k+1)}$. It is not difficult to see that the root can be represented as $\rho_{k/(k+1)} = \rho_1 + \varepsilon_k$ for some positive ε_k such that $0 < \rho_1 + \varepsilon_k < 1$. Note that $\varepsilon_k \to 0$ as k grows. Next, we substitute $\rho_1 + \varepsilon_k$ for x in $x + x^2 - 1 = x^{2k+3}$, use $1 = \rho_1 + \rho_1^2$ and obtain the following:

$$\begin{aligned}
&\rho_1 + \varepsilon_k + (\rho_1 + \varepsilon_k)^2 - 1 = (\rho_1 + \varepsilon_k)^{2k+3},\\
&\rho_1 + \varepsilon_k + \rho_1^2 + 2\rho_1\varepsilon_k + \varepsilon_k^2 - 1 = \rho_1^{2k+3}(1 + o(1)),\\
&\varepsilon_k(1 + 2\rho_1 + \varepsilon_k) = \rho_1^{2k+3}(1 + o(1)).
\end{aligned}$$

The claimed result $\rho_{k/(k+1)} = \rho_1 + C\rho_1^{2k}(1 + o(1))$ follows, because $1 + 2\rho_1 + \varepsilon_k \to 1 + 2\rho_1$ as $k \to \infty$. □

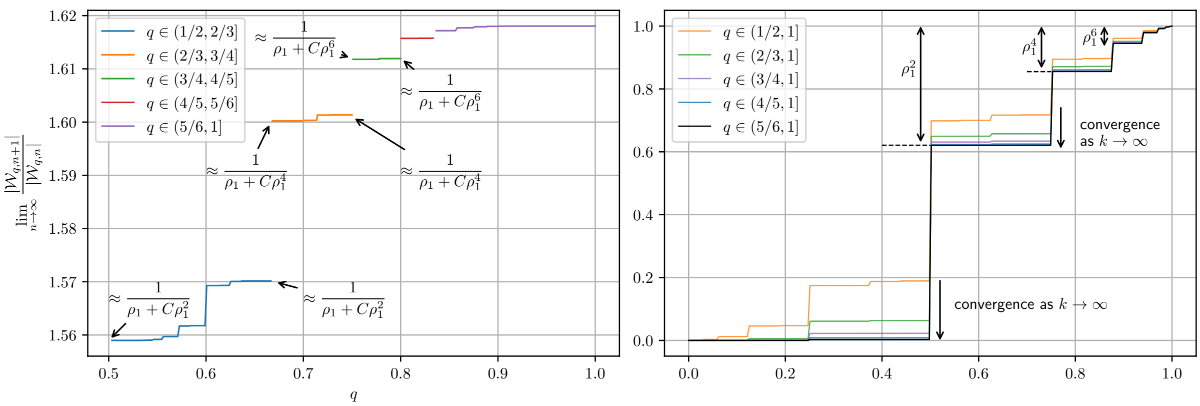


Figure 4: Fractal structure of the function $\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$, before and after rescaling on the intervals $(\frac{k-1}{k}, 1]$.

Using the Taylor expansion several times one can improve the root approximation and get $\rho_{k/(k+1)} = \rho_1 + C\rho_1^{2k} + O(k\rho_1^{4k})$. But in context of this paper, Proposition 5 is sufficient.

Proposition 6. *For natural $k \geqslant 2$, the smallest in modulus root $\rho^+_{(k-1)/k}$ of $\Pi^+_{(k-1)/k}$ is*

$$\rho_1 + C\rho_1^{2k}(1 + o(1)),$$

where $C = \rho_1^3/(1 + 2\rho_1)$, and ρ_1 is the smallest in modulus root of $1 - x - x^2$.

Proof. After arithmetic transformations similar to the ones in Proposition 5, the equation $\Pi^+_{(k-1)/k} = 0$, i.e. the equation

$$1 - x^{2k-1} - \sum_{i=0}^{k-2} x^{1+i+\left\lfloor \frac{ik}{k-1} \right\rfloor} - x^{2k-1} + x^{2k} = 0$$

turns into

$$x + x^2 - 1 = x^{2(k-1)}(2x^3 - x - x^4 + x^2).$$

Note that, at some point, we multiply both sides of equation by $(1 - x^2)$, adding -1 and 1 as roots. This does not change the picture dramatically, because $0 < \rho^+_{(k-1)/k} < 1$.

As in the proof of Proposition 5 the root is $\rho_1 + \varepsilon_k$, $\varepsilon_k \to 0$ when $k \to \infty$. Note that $2x^3 - x - x^4 + x^2 - x^5 = (x^3 - x)(1 - x - x^2)$, so we have $2\rho_1^3 - \rho_1 - \rho_1^4 + \rho_1^2 = \rho_1^5$, so $2(\rho_1 + \varepsilon_k)^3 - (\rho_1 + \varepsilon_k) - (\rho_1 + \varepsilon_k)^4 + (\rho_1 + \varepsilon_k)^2 = \rho_1^5(1 + o(1))$. Using the techniques from the previous proof we obtain the claimed result. □

Note that Propositions 5 and 6 already provide some insights into the fractal structure of $\Phi(q)$ displayed in Figure 4. On the left side of this figure, the horizontal axis is partitioned into intervals $(1/2, 2/3], (2/3, 3/4], \ldots$, and the parts of the plot of $\Phi(\cdot)$ are grouped accordingly. In particular, we showed that

$$\lim_{k\to\infty} \frac{\Phi(1) - \lim_{\delta\to 0^+} \Phi(\frac{k-1}{k} + \delta)}{\Phi(1) - \Phi(\frac{k}{k+1})} = 1,$$

i.e. that the images of the intervals $(\frac{k-1}{k}, \frac{k}{k+1}]$ tend to straighten as $k \to \infty$.

To better observe the self-similarity, the intervals $q \in (k/(k+1), 1]$ and their images under $\Phi(\cdot)$ can be "normalized" using *simple rescaling* and *Minkowski's question-mark function* [8, 17]. Simple rescaling takes a set of positive values V, containing at least 2 values, and maps every $v \in V$ to $\frac{v - \min V}{\max V - \min V}$, so the image lies in $[0, 1]$. Minkowski's question-mark function is a little trickier, and we must first discuss mediants and the construction of the Stern–Brocot tree [6, 19].

For two irreducible fractions a/b and c/d their *mediant* is defined as $(a + c)/(b + d)$. The root of the Stern–Brocot tree is $1/1$, which is the mediant of two conventionally irreducible fractions $1/0$ and $0/1$. To determine the left (resp. right) child of a node x/y of the level i we need to find the greatest (resp. smallest) fraction $x'/y' < x/y$ (resp. $x'/y' > x/y$) that appears in set of values of first i levels together with $1/0$ and $0/1$, and compute the mediant $(x + x')/(y + y')$. For instance, the left child of $2/3$ is $3/5$, it is calculated as the mediant of $1/2$ and $2/3$. Figure 5 illustrates this process.

Minkowski's question-mark function, denoted by $?(x)$, maps a positive rational value x to a positive dyadic rational $a/2^k$ with $a, k \in \mathbb{N}$. By definition, $?(0) = 0$ and $?(1) = 1$. Whenever $x \in (0, 1)$ is a rational number represented by an irreducible fraction a/b, such that in the Stern–Brocot tree it is constructed via taking a mediant of two fractions p/q and p'/q', its image under Minkowski's function is defined as

$$?\,(x) = ?\left(\frac{p + p'}{q + q'}\right) := \frac{1}{2}\left(?\left(\frac{p}{q}\right) + ?\left(\frac{p'}{q'}\right)\right).$$

In other words, we descend the Stern–Brocot tree in search of the a/b, and "in parallel" construct a resulting value by applying the mean instead of the mediant. For $x > 1$, Minkowski's function is defined as $?(x + 1) = ?(x) + 1$. In general, $?(x)$ is monotonically increasing, and can be defined on all $\mathbb{R}^+$ [8].

The right side of Figure 4 is obtained by applying the simple rescaling on the vertical axis and Minkowski's question-mark function followed by the simple rescaling on the horizontal axis for intervals $(k/(k + 1), 1]$ and their images. The similar analysis can be

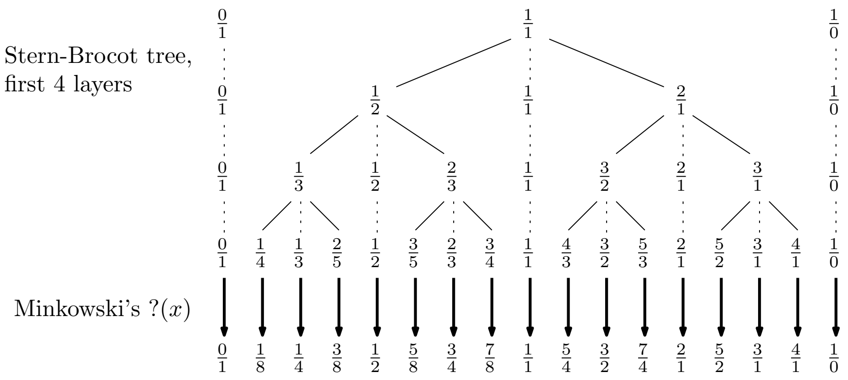


Figure 5: The Stern–Brocot tree and Minkowski's $?(x)$.

done for intervals $(1, (k+1)/k]$. The fractal structure of Φ presented in Figure 2 appears more regular in Figure 7 as we apply Minkowski's question-mark function over the x-axis.

Now, we study the more general case, that is the fractal structure of $\Phi(q)$ on the intervals $\left(\frac{p+ck}{q+dk}, \frac{c}{d}\right]$ for $k \geqslant 1$, where fractions are irreducible, and $\frac{p+c}{q+d}$ is the left child of c/d in the Stern–Brocot tree. For example the intervals $\left(\frac{3+5k}{2+3k}, \frac{5}{3}\right]$, where $8/5$ is the left child of $5/3$.

Proposition 7. *For natural $k \geqslant 1$, the smallest in modulus root of $\Pi_{\frac{p+ck}{q+dk}}$ is*

$$\rho_{\frac{p+ck}{q+dk}} = \rho_{c/d} + C\rho_{c/d}^{(c+d)k}\big(1 + o(1)\big),$$

where $k \geqslant 1$, $\frac{p+c}{q+d}$ is the left child of c/d in the Stern–Brocot tree, C is a constant depending only on p/q and c/d, and $\rho_{c/d}$ is the smallest in modulus root of $\Pi_{c/d}$.

Proof. Recall that from Proposition 4 we have

$$\Pi_{\frac{p+ck}{q+dk}} := 1 - x^{p+q+(c+d)k} - \sum_{i=0}^{p+ck-1} x^{1+i+\left\lfloor \frac{i(q+dk)}{p+ck} \right\rfloor}. \tag{3}$$

For $0 < i < p + ck$, $1 + i + \left\lfloor \frac{i(q+dk)}{p+ck} \right\rfloor$ equals the number of integer points with coordinates (i, y) lying between two diagonal lines intersecting at the origin with respective slopes $(-c)$ and $\frac{q+dk}{p+ck}$, see Figure 6 for an illustration.

Note that from the construction of Stern–Brocot tree, $\frac{p}{q} < \frac{c}{d}$. From the mediant inequality it follows that $\frac{p}{q} < \frac{p+ck}{q+dk} < \frac{c}{d}$, for any integer $k > 0$. Next, let us show that there are no integer points with horizontal coordinate equal $i, 0 < i < p+ck$, lying strictly between the lines with slopes $\frac{q+dk}{p+ck}$ and $\frac{d}{c}$. Consider the triangle ABC with following integer coordinates:

$$\begin{aligned} A &= (0, 0); \\ B &= (c(k+1), d(k+1)); \\ C &= (p + ck, q + dk). \end{aligned}$$

The area of ABC is

$$\frac{1}{2}\begin{vmatrix} c(k+1) & p+ck \\ d(k+1) & q+dk \end{vmatrix} = \frac{1}{2}(k+1)\begin{vmatrix} c & p \\ d & q \end{vmatrix} = \frac{k+1}{2}(cq - dp) = \frac{k+1}{2},$$

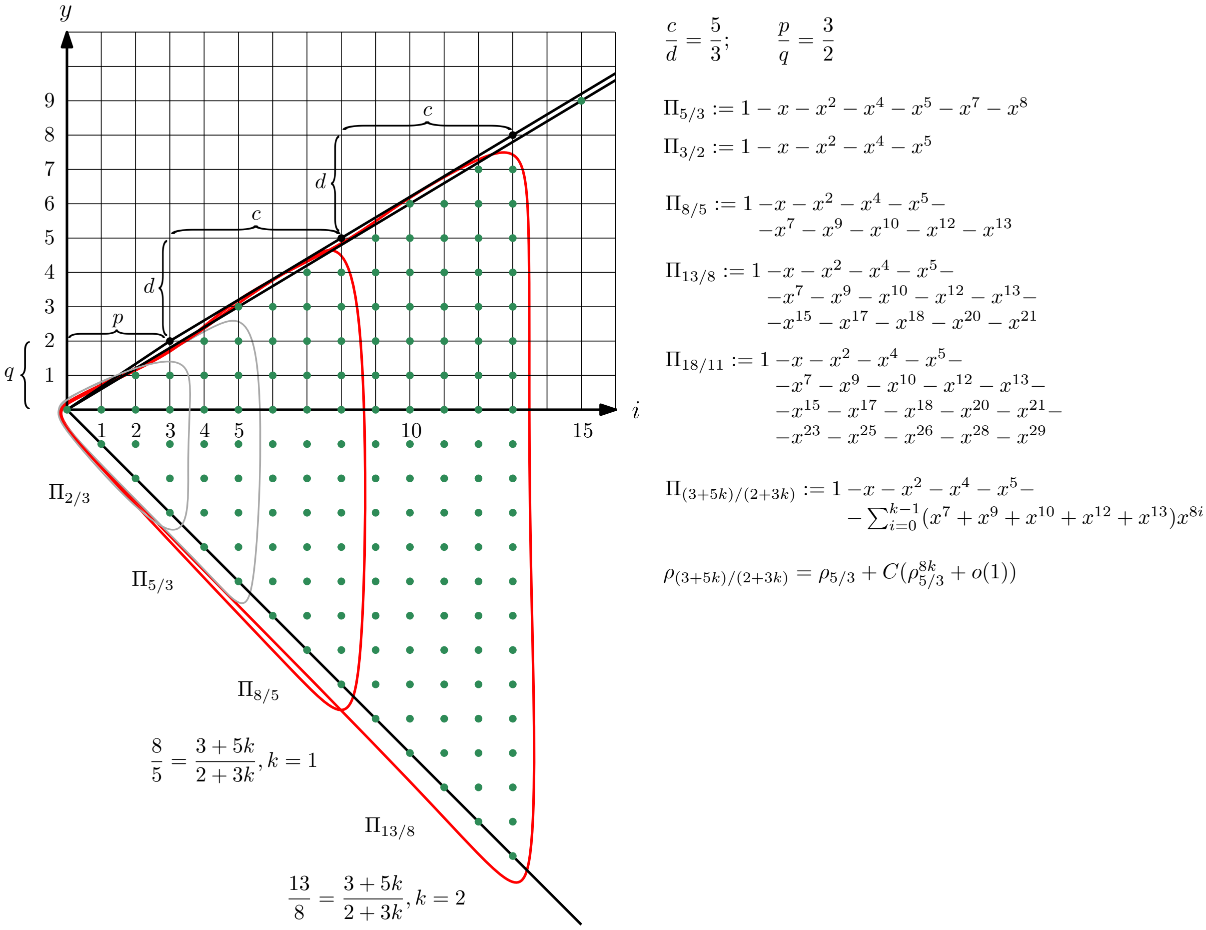


Figure 6: Geometric representation of polynomials $\Pi_{5/3}, \Pi_{3/2}$ and $\Pi_{(3+5k)/(2+3k)}$ for some values of $k \geqslant 1$.

because $\frac{p}{q} < \frac{c}{d}$ and $(cq - dp) = 1$ by a joli property of the Stern–Brocot tree (see [12]). Pick's Theorem [18] implies that the area of triangle ABC is equal to $I + B/2 - 1$, where I is the number of its interior points and exactly B points lie on the boundary. We have $B = k + 3$, because $\frac{p+ck}{q+dk}$ and $\frac{c}{d}$ are irreducible fractions. We conclude that $I = 0$ and there are no interior points inside ABC. This fact can be used to simplify $\Pi_{\frac{p+ck}{q+dk}}$ from Equation (3) by considering only points lying on or below the line of slope d/c and on or above the line of slope $(-c)$. Taking this into account, we rewrite the equation $\Pi_{\frac{p+ck}{q+dk}} = 0$, i.e. the equation

$$1 - x^{p+q+(c+d)k} - \sum_{i=0}^{p+ck-1} x^{1+i+\left\lfloor \frac{i(q+dk)}{p+ck} \right\rfloor} = 0$$

by decomposing the internal sum representing the sum over all points in a triangle (see Figure 6) by the sum over corresponding triangles and rectangles, which then allows us to simplify the sum by using the summation formula for geometric progressions:

$$1-\sum_{j=0}^{p} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} - \sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} \sum_{i=0}^{k-1} x^{(c+d)i} = 0, \tag{4}$$

$$1-\sum_{j=0}^{p} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} - \sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} \frac{1-x^{(c+d)k}}{1-x^{c+d}} = 0,$$

$$1-\sum_{j=0}^{p} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} - x^{c+d} + x^{c+d}\sum_{j=0}^{p} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} - \sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} + x^{(c+d)k}\sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} = 0,$$

$$1-x^{c+d}-\sum_{j=0}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} + \sum_{j=c}^{p+c} x^{1+j+d+\left\lfloor \frac{(j-c)d}{c}\right\rfloor} + x^{(c+d)k}\sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} = 0,$$

$$1-x^{c+d}-\sum_{j=0}^{c-1} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} + x^{(c+d)k}\sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor} = 0. \tag{5}$$

At some point, we multiply both sides by $(1-x^{c+d})$. All roots of Equation (4) are also the roots of Equation (5), but the latter is additionally satisfied by the roots of unity $1 = x^{c+d}$, the modulus of which is greater than $\rho_{\frac{p+ck}{q+dk}}$.

Finally, since the first three terms of the sum are equal to $\Pi_{c/d}(x)$, we conclude that for $x = \rho_{\frac{p+ck}{q+dk}}$ we have

$$-\Pi_{c/d}(x) = x^{(c+d)k}\sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor}.$$

Using the same method as in the proof of Proposition 5 and denoting by $\Pi'_{c/d}$ the derivative of $\Pi_{c/d}$ we obtain

$$\rho_{\frac{p+ck}{q+dk}} = \rho_{c/d} + C\rho_{c/d}^{(c+d)k}\big(1+o(1)\big),$$

where

$$C = \frac{\sum_{j=p+1}^{p+c} \rho_{c/d}^{1+j+\left\lfloor \frac{jd}{c}\right\rfloor}}{-\Pi'_{c/d}\left(\rho_{c/d}\right)}.$$

□

Proposition 8. *For natural $k \geqslant 2$, the smallest in modulus root $\rho^{+}_{\frac{p+c(k-1)}{q+d(k-1)}}$ of $\Pi^{+}_{\frac{p+c(k-1)}{q+d(k-1)}}$ is*

$$\rho_{\frac{p+ck}{q+dk}} = \rho_{c/d} + C\rho_{c/d}^{(c+d)k}\big(1+o(1)\big),$$

where $k \geqslant 1$, $\frac{p+c}{q+d}$ is the left child of c/d in the Stern–Brocot tree, C is the same constant as in Proposition 7 not depending on k, and $\rho_{c/d}$ is the smallest in modulus root of $\Pi_{c/d}$.

Proof. Recall that Proposition 4 defines $\Pi^{+}_{a/b} := \Pi_{a/b} - x^{a+b} + x^{a+b+1}$. Having this in mind and adapting the equations from the proof of Proposition 7 by writing $k-1$ in place of k we obtain:

$$1 - x^{c+d} - \sum_{j=0}^{c-1} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} + x^{(c+d)(k-1)} \sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} - \\ -x^{p+q+(c+d)(k-1)}(1-x^{c+d}) + x^{p+q+(c+d)(k-1)+1}(1-x^{c+d}) = 0,$$

$$1 - x^{c+d} - \sum_{j=0}^{c-1} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} + x^{(c+d)(k-1)} \times \\ \times \left(\sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} - x^{p+q} + x^{p+q+(c+d)} + x^{p+q+1} - x^{p+q+(c+d)+1} \right) = 0. \tag{6}$$

Now, using the fact that $cq - dp = 1$ and a kind of geometrical argument (as in the previous proof), it can be shown that

$$\sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} - x^{p+q} + x^{p+q+(c+d)} + x^{p+q+1} - x^{p+q+(c+d)+1} - x^{(c+d)} \sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} = \\ = \left(1 - x^{c+d} - \sum_{j=0}^{c-1} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor}\right) \left(x^{c+d+p+q} - x^{p+q}\right).$$

Adapting the techniques from the proofs of Propositions 5 and 6 we see that the smallest in modulus root of (6) is equal to the smallest in modulus root of

$$1 - x^{c+d} - \sum_{j=0}^{c-1} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} + x^{(c+d)(k-1)} \left(x^{(c+d)} \sum_{j=p+1}^{p+c} x^{1+j+\left\lfloor \frac{jd}{c} \right\rfloor} + o(1) \right) = 0,$$

where $o(1)$ is considered as $k \to \infty$. Comparing it to Equation (5) from the proof of Proposition 7, we see that the claimed result follows. □

To summarize the results of the previous two propositions, let q denote $\frac{c}{d}$ and let q_k^* denote $\frac{p+ck}{q+dk}$, i.e. the kth left approximation of q in the Stern–Brocot tree. We are essentially showing that the portrait of $\Phi(q)$ rescaled on the intervals $(q_k^*, q]$ tends to a constant function on the semi-open intervals $(q_{k-1}^*, q_k^*]$:

$$\lim_{k\to\infty} \frac{\Phi(q) - \lim_{\delta \to 0^+} \Phi(q_{k-1}^* + \delta)}{\Phi(q) - \Phi(q_k^*)} = 1$$

and also that the ratio between consecutive constants on the rescaled picture tends to ρ_q^{c+d}:

$$\lim_{k\to\infty} \frac{\Phi(q) - \Phi(q_k)}{\Phi(q) - \Phi(q_{k-1})} = \rho_q^{c+d}.$$

Using a similar technique, it is possible to demonstrate an identical picture using the right child of the Stern–Brocot tree, for example, for the intervals of the form $[1, \frac{k+1}{k})$, and, more generally, for all rational points $q = c/d$.

The sequence of positive rational numbers ordered by corresponding jumps of the function $\Phi(q) = \lim_{n\to\infty} |\mathcal{W}_{q,n+1}|/|\mathcal{W}_{q,n}|$ (see Proposition 4) starts with

$$1, \frac{1}{2}, 2, \frac{1}{3}, \frac{1}{4}, 3, \frac{2}{3}, \frac{1}{5}, \frac{1}{6}, \frac{3}{2}, \frac{1}{7}, 4, \frac{2}{5}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{3}{4}, \frac{1}{11}, \frac{2}{7}, \frac{1}{12}, 5, \frac{3}{5}, \frac{1}{13}, \frac{4}{3}, \frac{1}{14}, \frac{2}{9}, \frac{1}{15}, \frac{1}{16}, \frac{5}{2}, \frac{1}{17}, \ldots$$

Question: Is it possible to explain this sequence without polynomial root calculations?

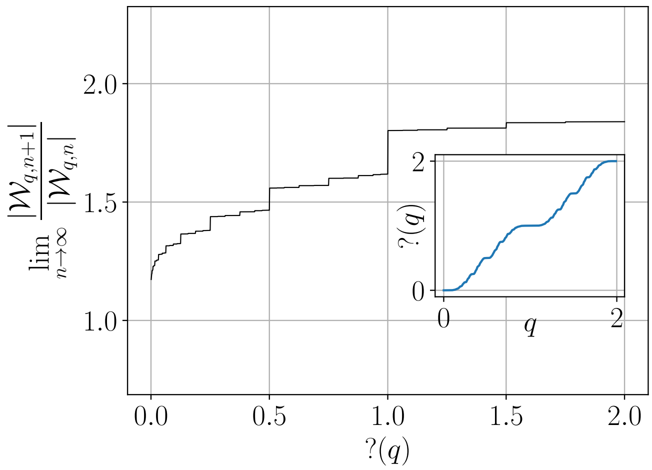


Figure 7: $\Phi(q) := \lim_{n\to\infty} |\mathcal{W}_{q,n+1}|/|\mathcal{W}_{q,n}|$ as a function of $?(q)$.

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