$\mathbb{Q}$-bonacci words and numbers

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Overview

- Classical Fibonacci words
- $\mathbb{Q}$-bonacci words
- Fractionally generalized golden ratio

In this talk Fibonacci words should not be confused with Sturmian Fibonacci words.

Initial terms: 0,...,0,0,1,

$a_n = a_{n-1} + a_{n-2}$, Fibonacci

$b_n = b_{n-1} + b_{n-2} + b_{n-3}$, Tribonacci

$c_n = c_{n-1} + c_{n-2} + c_{n-3} + c_{n-4}$, Tetranacci

- Generalized Fibonacci numbers and associated matrices, 1960
  E. P. Miles Jr.

- Fibonacci-Tribonacci, 1963
  M. Feinberg

Classical Fibonacci words

Words avoiding 1^k are counted by generalized k-step Fibonacci numbers.

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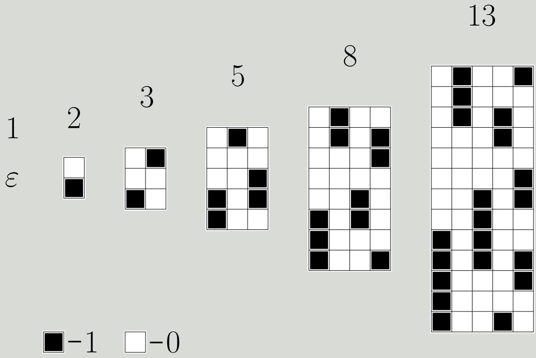
Words avoiding 11 are counted by Fibonacci.

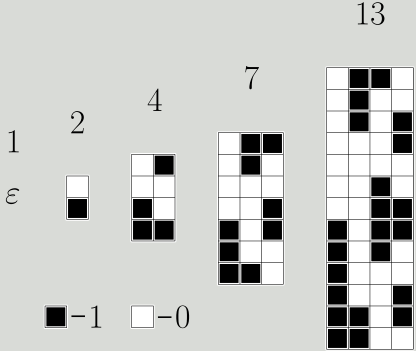


Words are listed in Gray order (consecutive differ in only 1 position)

Classical Fibonacci words

Words avoiding 1^k are counted by generalized k-step Fibonacci numbers. ***Hi José L. Ramírez ;)***

Words avoiding 111 are counted by Tribonacci



Words are listed in Gray order (consecutive differ in only 1 position)

Words avoiding 1^k are counted by generalized Fibonacci numbers

Let $\mathcal{B}_n(1^k)$ be the set of binary words of length n avoiding 1^k,

$$|\mathcal{B}_n(1^k)| = f_{n+k,k},$$

where $f_{n,k}$ is a generalized Fibonacci number defined as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \le n \le k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases}$$

$f_{n,2}$: Fibonacci
$f_{n,3}$: Tribonacci
$f_{n,4}$: Tetranacci

Classical Fibonacci words literature

- The Art of Computer Programming, Vol. 3: Sorting and Searching, 2 ed. (page 286), 1998, Donald Knuth
- Matters Computational (Section 14.2), 2010, Jörg Arndt
  https://www.jjj.de/fxt/fxtbook.pdf
- Combinatorial Gray codes-an updated survey, 2022
  Torsten Mütze, https://arxiv.org/pdf/2202.01280.pdf
- Generalized Fibonacci cubes are mostly Hamiltonian
  Jenshiuh Liu, Wen-Jing Hsu, Moon Jung Chung, 1994
- Gray codes for A-free strings. Matthew B. Squire, 1996
- A loopless generation of bitstrings without p consecutive ones
  Vincent Vajnovszki, 2001
- An O(1) time algorithm for generating Fibonacci strings
  Kenji Mikawa and Ishiro Semba, 2005
- Counting on Fibonacci Polyominoes and Fibonacci Graphs
  José L. Ramírez, 2022, This Fibonacci Conference :)

Can we extend
the definition of $f_{n,k}$
to cover the case where k is
not an integer?

Half-bonacci numbers?

Yes, we can!
Let’s see how!

Definition

An n-length binary word is *q-decreasing*, $q \in \mathbb{N}^+$, if every of its length maximal factors of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$.

$$\cdots 1 \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} 0 \cdots$$

Let $\mathcal{W}_{q,n}$ be the set of such words of length n.
Let $\mathcal{W}_q = \bigcup_{n \in \mathbb{N}} \mathcal{W}_{q,n}$.

1-decreasing words, $\mathcal{W}_1$

In particular, in a 1-decreasing word every run of 0s is immediately followed by a strictly shorter run of 1s.

$$\cdots 1 \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} 0 \cdots \qquad a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	3	5	8	Fibonacci

0	00	000	0000	…
1	10	001	0001	
	11	100	0010	
		110	1000	
		111	1001	
			1100	
			1110	
			1111	

2-decreasing words, $\mathcal{W}_2$

$$\cdots 1\,\underbrace{000\cdots\cdots 00}_{a}\,\underbrace{111\cdots 11}_{b}\,0\cdots \qquad \text{where } 2a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	⋯
	2	4	7	13	Tribonacci

			0000	
			0001	
			0010	
		000	0011	
		001	0100	
	00	010	0101	
0	01	100	1000	...
1	10	101	1001	
	11	110	1010	
		111	1100	
			1101	
			1110	
			1111	

q-decreasing words with natural q

- Bijections between q-decreasing words and words avoiding factors 1^{q+1}.
- Efficient generation and Gray codes
- Solved Eğecioğlu-Iršič conjecture (Hamiltonian path always exists in Fibonacci-run graphs)
- Mean bit value in random words

q-decreasing Fibonacci words literature

- Gray codes for Fibonacci q-decreasing words
  Jean-Luc Baril, Sk and Vincent Vajnovszki
  https://arxiv.org/abs/2010.09505
  Theoretical Computer Science, 2022.
- Fibonacci-run graphs I: Basic properties
  Ömer Eğecioğlu and Vesna Iršič
  https://arxiv.org/abs/2010.05518
  Discrete Applied Mathematics, 2021
- Qubonacci words, BKV
  Presented at Permutations patterns 2021
  https://kirgizov.link/talks/qubonacci.pdf
- Asymptotic bit frequency in Fibonacci words, BKV
  Presented at GASCom 2022
  https://kirgizov.link/talks/gascom2022.pdf
  https://arxiv.org/abs/2106.13550
  Pure Mathematics and Applications, 2022

From $\mathbb{N}$ to $\mathbb{Q}^+$

Definition II

An n-length binary word is *q-decreasing*, $q \in \mathbb{N}^+$, if every of its length maximal factors of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$.

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1/2-decreasing words (half-bonacci)

$$\cdots 1 \underbrace{000\cdots\cdots 00}_{a} \underbrace{111\cdots 11}_{b} 0 \cdots \qquad \text{where } \frac{1}{2}a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	5	$\cdots$
	2	3	4	6	9	Narayana's cows, year 1356

$a_n = a_{n-1} + a_{n-3}$

0	00	000	0000	00000	...
1	10	100	0001	00001	
	11	110	1000	00010	
		111	1100	10000	
			1110	10001	
			1111	11000	
				11100	
				11110	
				11111	

Construction

Example $q = 1/2$

Every word from $\mathcal{W}_{1/2}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0

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0 **<u>001</u>**

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000 $\underline{\mathbf{001}}$ 1

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$00000\,\underline{\mathbf{001}}\,11$

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000 **<u>001</u>** 1
00000 **<u>001</u>** 11
0000000 **<u>001</u>** 111

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$\cdots$

$$\overbrace{0\cdots00}^{1+2i \text{ zeros}}\ \underbrace{1\cdots11}_{i \text{ ones}}$$

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$\cdots$
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Model polynomial $P_{1/2}(y, z) = z$ encodes the initial admissible suffix 0.

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0 **001**
000 **001** 1
00000 **001** 11
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000000000 **001** 1111
…

$$\overbrace{0\cdots00}^{1+2i \text{ zeros}}\ \underbrace{1\cdots11}_{i \text{ ones}}$$

Model polynomial $P_{1/2}(y, z) = z$ encodes the initial admissible suffix 0.

Spawning infix 001 is encoded by $z^2 y$.

Admissible suffixes are constructed iteratively by injecting the spawning infix 001 just after the last 0 in already constructed suffixes.

Example $q = 2/3$

Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1 \, \sigma_1 \, \sigma_2 \, \cdots \, \sigma_\ell \, ,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001

Example $q = 2/3$

Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001
0 **<u>00011</u>**

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Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1 \, \sigma_1 \, \sigma_2 \, \cdots \, \sigma_\ell \, ,$$

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Admissible suffixes
0
001
0 **00011**
00 **00011** 1

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Admissible suffixes
0
001
0 **00011**
00 **00011** 1
0000 **00011** 11

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Admissible suffixes
0
001
0 **<u>00011</u>**
00 **<u>00011</u>** 1
0000 **<u>00011</u>** 11
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Admissible suffixes
0
001
0 **00011**
00 **00011** 1
0000 **00011** 11
00000 **00011** 111

$$\overbrace{0\cdots00}^{1+\lfloor \frac{i}{q} \rfloor \text{ zeros}} \quad \underbrace{1\cdots11}_{i \text{ ones}}$$

Model polynomial $P_{1/2}(y,z) = z + z^2 y$ encodes initial admissible suffixes 0 and 001.

Spawning infix 00011 is encoded by $z^3 y^2$.

Admissible suffixes are constructed iteratively by injecting the spawning infix 000111 just after the last 0 in already constructed suffixes.

q	Model polynomial	Spawning infix g.f.
$1/k$	z	$z^k y$
2	$z + zy$	zy^2
$2/3$	$z + z^2y$	z^3y^2
$3/2$	$z + zy + z^2y^2$	z^2y^3
$3/4$	$z + z^2y + z^3y^2$	z^4y^3
$3/5$	$z + z^2y + z^4y^2$	z^5y^3
...	...	...

Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$.

Spawning infix, $\underbrace{0\cdots00}_{d}\underbrace{11\cdots1}_{c}$, has g.f. $z^d y^c$.

Model polynomial is $P_{q=\frac{c}{d}}(y,z) = \sum_{i=0}^{c-1} z^{1+\lfloor \frac{i}{q} \rfloor} y^i$.

Generating function

Theorem 1

Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$. The generating function

$$W_q(y, z) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} w_{r,i} z^r y^i$$

where $w_{r,i}$ is number of words from $\mathcal{W}_q$ of length $r + i$ containing exactly r zeros and i ones is

$$W_q(y, z) = \frac{1 - z^d y^c}{(1 - y)\big(1 - z^d y^c - P_q(y, z)\big)},$$

where $P_q(y, z)$ is the model polynomial of q.

Linear recurrence with 0–1 coefficients

Theorem 2

Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$. The number of n-length binary words from $\mathcal{W}_{q,n}$, denoted by w_n, can be expressed as

$$w_n = \sum_{j \in J} w_{n-j} + w_{n-(c+d)}, \tag{1}$$

where J is the set of powers from $P_q(x,x)$.

For example, when $q = \frac{3}{2}$,
we have $P_{\frac{3}{2}}(x,x) = x + x^2 + x^4$, and $J = \{1, 2, 4\}$.

With appropriate initial conditions:)

q	Sequence	Recurrence relation	OEIS (with shifts)
1/5	1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, ...	$w_n = w_{n-1} + w_{n-6}$	Compositions (ordered partitions) of n into 1s and 6s. A5708
1/4	1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, ...	$w_n = w_{n-1} + w_{n-5}$	C. into 1s and 5s. A3520
1/3	1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, ...	$w_n = w_{n-1} + w_{n-4}$	C. into 1s and 4s. A3269
2/5	1, 2, 3, 4, 6, 9, 13, 18, 26, 38, 55, 79, ...	$w_n = w_{n-1} + w_{n-4} + w_{n-7}$	C. into 1s, 4s and 7s. Not in OEIS.
1/2	1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, ...	$w_n = w_{n-1} + w_{n-3}$	Narayana's cows, A930
3/5	1, 2, 3, 5, 8, 12, 19, 30, 46, 72, 113, 176, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-6} + w_{n-8}$	NEW
2/3	1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5}$	C. into 1s, 3s and 5s, A60961
3/4	1, 2, 3, 5, 8, 13, 21, 33, 53, 85, 136, 218, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5} + w_{n-7}$	C. into 1s, 3s, 5s and 7s, A117760
4/5	1, 2, 3, 5, 8, 12, 19, 30, 46, 72, 113, 176, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5} + w_{n-7} + w_{n-9}$	NEW
1	1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...	$w_n = w_{n-1} + w_{n-2}$	Fibonacci numbers, A45
5/4	1, 2, 4, 7, 13, 23, 42, 75, 136, 244, 441, 794, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-6} + w_{n-8} + w_{n-9}$	NEW
4/3	1, 2, 4, 7, 13, 23, 42, 75, 136, 245, 443, 799, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-6} + w_{n-7}$	NEW
3/2	1, 2, 4, 7, 13, 23, 42, 76, 138, 250, 453, 821, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-5}$	NEW
5/3	1, 2, 4, 7, 13, 24, 44, 81, 148, 272, 499, 916, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-5} + w_{n-7} + w_{n-8}$	NEW
2	1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3}$	Tribonacci numbers, A73
5/2	1, 2, 4, 8, 15, 29, 56, 107, 206, 396, 761, 1463, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-5} + w_{n-6} + w_{n-7}$	NEW
3	1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4}$	Tetranacci numbers, A78
4	1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4} + w_{n-5}$	Pentanacci numbers, A1591
5	1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4} + w_{n-5} + w_{n-6}$	Hexanacci numbers, A1592

Generalized golden ratio

Consider the following function

$$q \mapsto \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$$

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For $q = 1$, we get the golden ratio ($\mathcal{W}_{1,n}$ is counted with the Fibonacci numbers).

Consider the following function

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For $q = 5/3$?

Consider the following function

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For $q = 1$, we get the golden ratio
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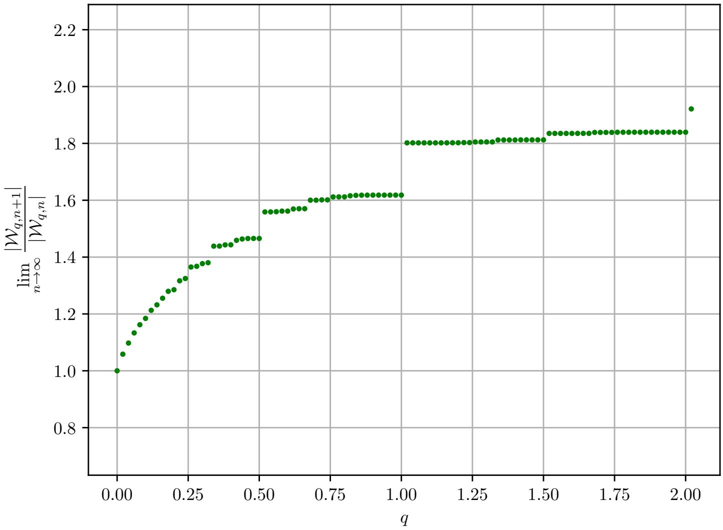
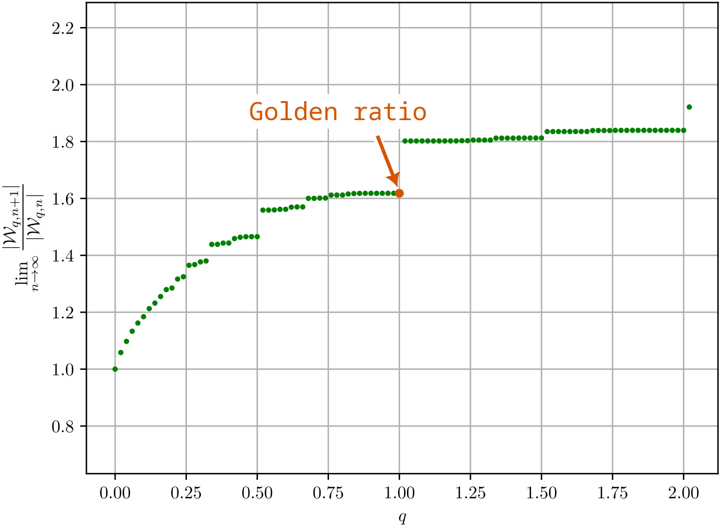
For $q = 2$, it is the tribonacci constant.
For $q = 5/3$?
For $q = \varphi$?

Fractional generalization of the golden ratio



$$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$$ as a function of q.

Fractional generalization of the golden ratio



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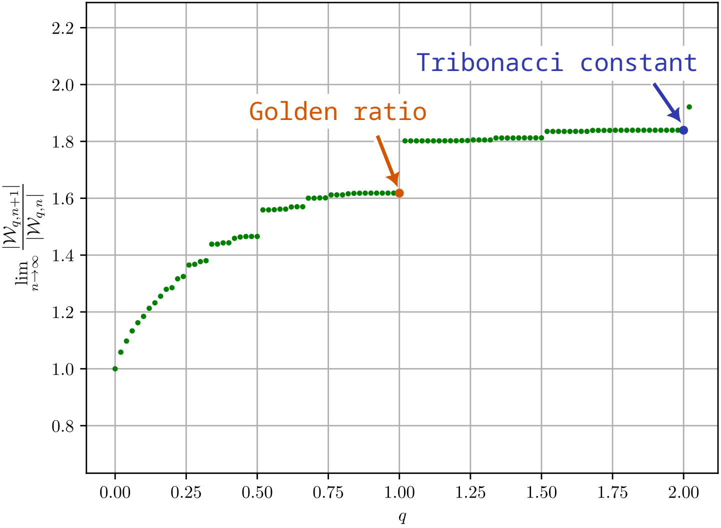

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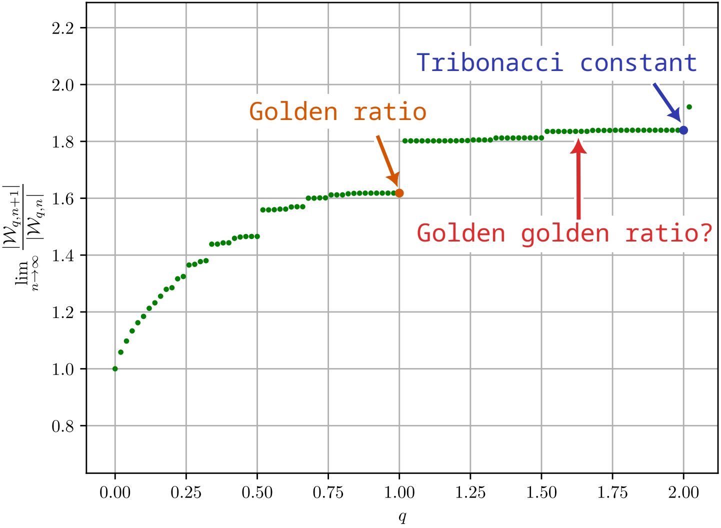

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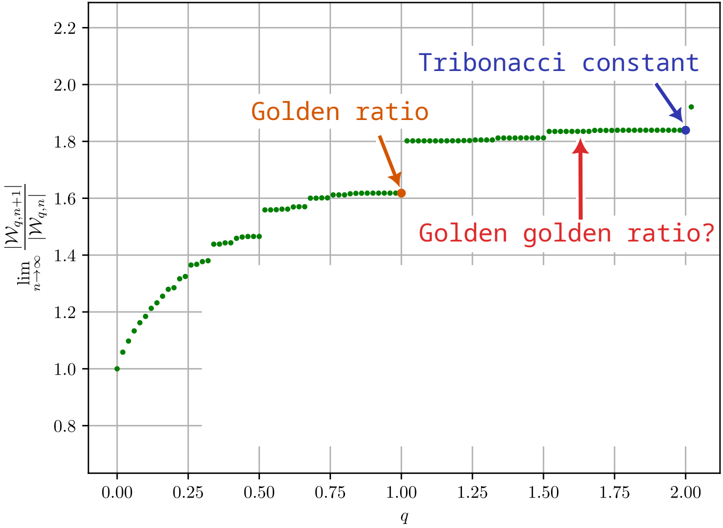

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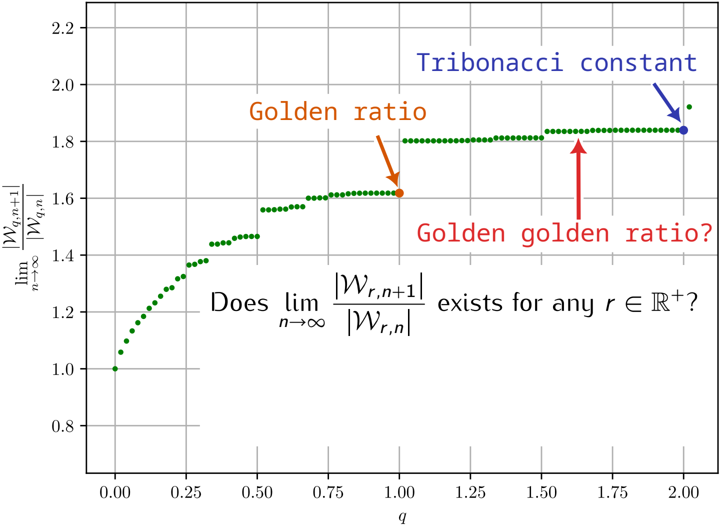

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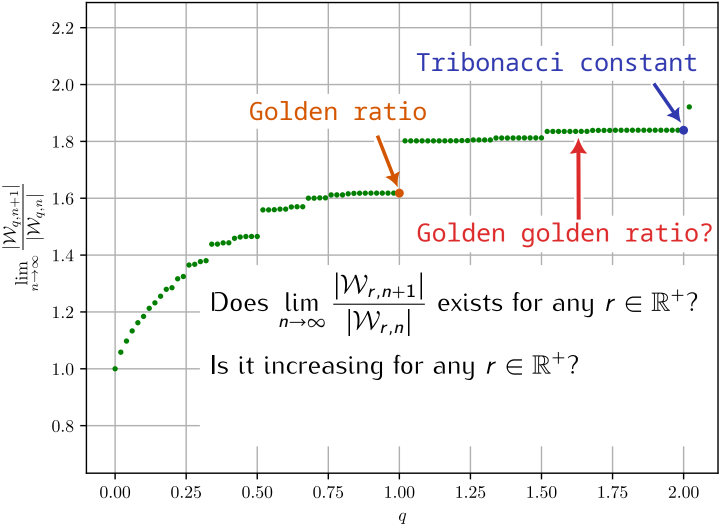
Fractional generalization of the golden ratio



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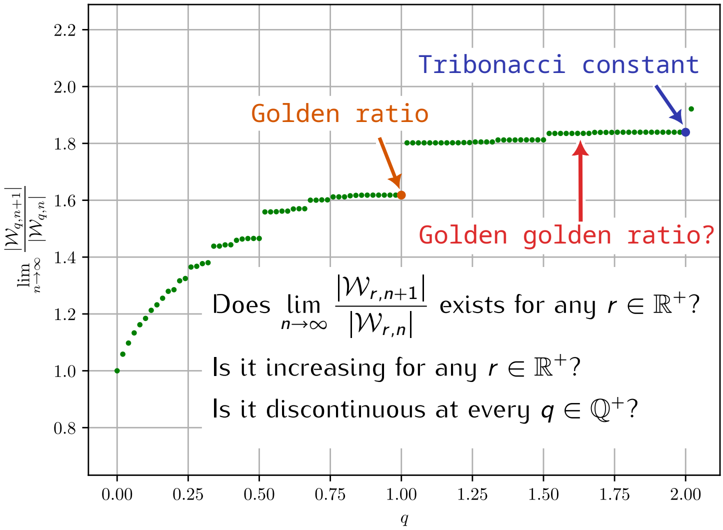

$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ as a function of q.

Conjectures

1. We conjecture the existence of a Gray code for $q \geq 2, q \in \mathbb{N}$. It is proven for $q = 1$.

2. For any $r \in \mathbb{R}^+$, $\lim_{n\to\infty} \frac{|\mathcal{W}_{r,n+1}|}{|\mathcal{W}_{r,n}|}$ exists.

3. The function $r \mapsto \lim_{n\to\infty} \frac{|\mathcal{W}_{r,n+1}|}{|\mathcal{W}_{r,n}|}$ is increasing.

4. This function is discontinuous at every $r \in \mathbb{Q}^+$.

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Permutation Patterns 2023

The International Conference on Permutation Patterns 2023 will take place at the University of Burgundy located in Dijon, where Gustave Eiffel was born, France, July 3-7, 2023.

The keynote speakers will be Torsten Mütze and TBD.

Program	Submission	Registration
Committees	Participants	Local Info
Travel to Dijon	Accommodations	Travel Support
Social	Booklet	Photos

A conference poster is available for download here: TBD.

More information about the conference series can be found at permutationpatterns.com.

The conference is supported by le conseil régional de Bourgogne-Franche-Comté, l'Université Bourgogne - Franche-Comté, Dijon Métropole, Università degli Studi di Firenze, le Laboratoire d'Informatique de Bourgogne, l'Agence Nationale de la Recherche (Project ANR Pics), National Science Foundation and National Security Agency.

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