Asymptotic bit frequency in Fibonacci words

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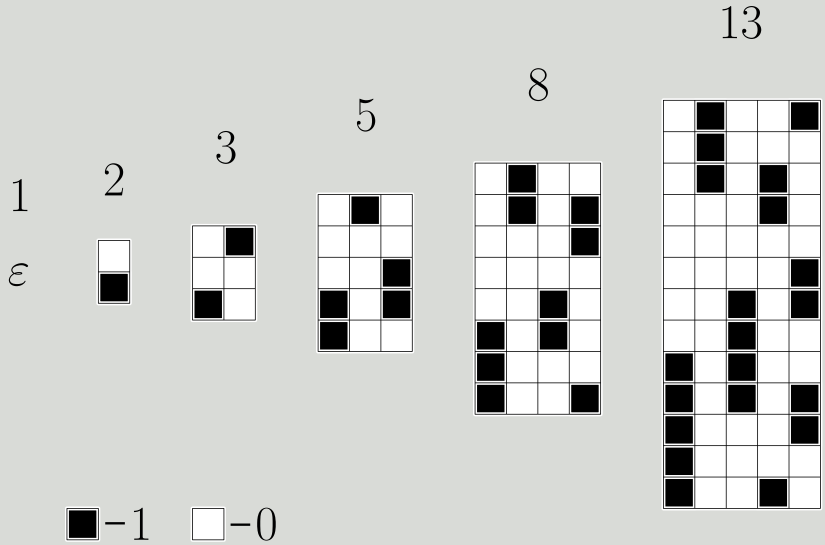
Overview

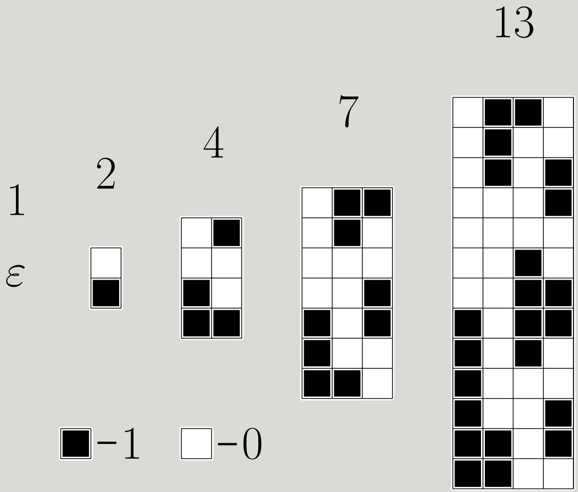
- Words avoiding k consecutive ones (Fibonacci words) counted by generalized Fibonacci sequence.
- Distribution of ones in these words
- Links to *q-decreasing words*, another kind of binary words counted by (generalized) Fibonacci sequence.

Our Fibonacci words should not be confused with Sturmian Fibonacci words

Words avoding 11 are counted by Fibonacci



Words avoding 111 are counted by Tribonacci



Words avoding 1^k are counted by generalized Fibonacci numbers

Let $\mathcal{B}_n(1^k)$ be the set of binary words of length n avoiding 1^k,

$$|\mathcal{B}_n(1^k)| = f_{n+k,k},$$

where $f_{n,k}$ is a generalized Fibonacci number defined as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \leq n \leq k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases}$$

Generalized Fibonacci numbers and associated matrices, 1960
E. P. Miles Jr.

Fibonacci-Tribonacci, 1963
Mark Feinberg

Fibonacci words and their Gray codes

- The Art of Computer Programming, Vol. 3: Sorting and Searching, 2 ed. (page 286), 1998, Donald Knuth
- Matters Computational (Section 14.2), 2010, Jörg Arndt
  https://www.jjj.de/fxt/fxtbook.pdf
- Combinatorial Gray codes-an updated survey, 2022
  Torsten Mütze, https://arxiv.org/pdf/2202.01280.pdf
- Generalized Fibonacci cubes are mostly Hamiltonian
  Jenshiuh Liu, Wen-Jing Hsu, Moon Jung Chung, 1994
- Gray codes for A-free strings. Matthew B. Squire, 1996
- A loopless generation of bitstrings without p consecutive ones
  Vincent Vajnovszki, 2001
- An O(1) time algorithm for generating Fibonacci strings
  Kenji Mikawa and Ishiro Semba, 2005

Distribution of ones

Construction. Example

Let $\mathcal{B}(1^k) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(1^k)$ be the set of binary words avoiding factors 1^k.

Example

$\mathcal{B}(111)$ contains the empty word, 1, and 11, and all other words from $\mathcal{B}(111)$ are constructed as

- $0w$,
- $10w$,
- $110w$,

where w is another word from $\mathcal{B}(111)$.

Construction. General case

Let $\mathcal{B}(1^k) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(1^k)$ be the set of binary words avoiding factors 1^k. It respects the following recursive decomposition

$$\mathcal{B}(1^k) = \mathbb{1}_{k-1} \cup \left(\bigcup_{i=0}^{k-1} \left(1^i 0 \cdot \mathcal{B}(1^k) \right) \right)$$

where $\mathbb{1}_{k-1} = \bigcup_{i=0}^{k-1} \{1^i\}$ is the set of words in $\mathcal{B}(1^k)$ containing no 0s, and $\cdot$ denotes the concatenation. The empty word also lies in $\mathbb{1}_{k-1}$.

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Now we can write an equation for the bivariate generating function

$$F_k(x, y) = \sum_{i=0}^{k-1} x^i y^i + F_k(x, y) \sum_{i=0}^{k-1} x^{i+1} y^i.$$

Distribution of ones in these words

Bivariate generating function

$$F_k(x,y) = \sum_{n,m \geq 0} a_{n,m} x^n y^m = \frac{1 - x^k y^k}{1 - xy - x + x^{k+1} y^k}$$

whose coefficient $a_{n,m}$ equals the number of words from $\mathcal{B}_n(1^k)$ containing exactly m 1s.

For example, when $k = 2$, we have

$m \backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2			1	3	6	10	15	21	28
3					1	4	10	20	35
4							1	5	15
5									1

- $P_k(x) = \frac{\partial F_k(x,y)}{\partial y}|_{y=1}$ is the generating function where the coefficient of x^n is the total number of 1s in $\mathcal{B}_n(1^k)$. We have

$$P_k(x) = \frac{x \cdot \sum_{i=0}^{k-2}(i+1)x^i}{\left(x^k + x^{k-1} + \cdots + x^2 + x - 1\right)^2}.$$

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- $T_k(x) = x\frac{\partial F_k(x,1)}{\partial x}$ is the generating function where the coefficient of x^n equals the total number of all bits in $\mathcal{B}_n(1^k)$. We have

$$T_k(x) = \frac{x\left(\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i\right)}{\left(x^k + x^{k-1} + \cdots + x^2 + x - 1\right)^2}.$$

- The expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$ is

$$\frac{[x^n]P_k(x)}{[x^n]T_k(x)}$$

.

Random bit in a random word

Theorem

The expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$, tends to μ_k, when $n \to \infty$, where

$$\mu_k = \left.\frac{kx^k - kx^{k-1} - x^k + 1}{kx^k - kx^{k-1} + x^{2k} - 3x^k + 2}\right|_{x=1/\varphi_k}$$

and $\varphi_k = \lim_{n\to\infty} f_{n+1,k}/f_{n,k}$ is the generalized golden ratio, in particular φ_2 is the golden ratio.

The limit of the expected bit value of binary words avoiding k consecutive 1s, whose length tends to infinity, approaches 1/2 as k grows:

$$\lim_{k\to\infty} \mu_k = \frac{1}{2}$$

Proof Ingredients.

- Classical asymptotic analysis, e.g. "Theorem 4.1" from

  An introduction to the analysis of algorithms, 2013
  Robert Sedgewick and Philippe Flajolet

- Irreducibility of the Fibonacci polynomial

$$x^k - x^{k-1} - \cdots - x^2 - x - 1$$

  See, for example, David Wolfram's paper

  Solving generalized Fibonacci recurrences, 1998

Links to q-decreasing words

Definition of q-decreasing words

An n-length binary word is *q-decreasing*, $q \in \mathbb{N}^+$, if every of its length maximal factors of the form 0^a1^b satisfies $a = 0$ or $q \cdot a > b$.

$$\cdots 1 \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} 0 \cdots$$

- Gray codes for Fibonacci q-decreasing words
  Jean-Luc Baril, Sk and Vincent Vajnovszki
  https://arxiv.org/abs/2010.09505
  To appear in Theoretical Computer Science.

- Qubonacci words, BKV
  Presented at Permutations patterns 2021
  https://kirgizov.link/talks/qubonacci.pdf

1-decreasing words

In particular, in a 1-decreasing word every run of 0s is immediately followed by a strictly shorter run of 1s.

$$\cdots 1 \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} 0 \cdots \qquad a > b$$

Let's count!

n	1	2	3	4	$\cdots$
	2	3	5	8	Fibonacci

0	00	000	0000	...
1	10	001	0001	
	11	100	0010	
		110	1000	
		111	1001	
			1100	
			1110	
			1111	

2-decreasing words

$$\cdots 1 \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} 0 \cdots \qquad \text{where } 2 \cdot a > b$$

Let's count!

n	1	2	3	4	...
	2	4	7	13	Tribonacci

0	00	000	0000	...
1	01	001	0001	
	10	010	0010	
	11	100	0011	
		101	0100	
		110	0101	
		111	1000	
			1001	
			1010	
			1100	
			1101	
			1110	
			1111	

Bit distribution in q-decreasing words

The bivariate generating function $W_q(x,y) = \sum_{n,m\geq 0} w_{n,m} x^n y^m$ where the coefficient $w_{n,m}$ is the number of n-length q-decreasing words containing exactly m 1s is given by:

$$W_q(x,y) = \frac{1 - x^{q+1}y^q}{1 - xy - x + x^{q+2}y^{q+1}}.$$

Recall that for 1^k avoiding words we have

$$F_k(x,y) = \frac{1 - x^k y^k}{1 - xy - x + x^{k+1}y^k}.$$

Take $k = q + 1$ and compare them...
They are not so different!

Mean bit

- In genereal we have more 1s in q-decreasing words than in words avoiding 1^{q+1}. Example:

	1-decreasing	avoiding 11
	000	000
	001	001
	100	010
	110	100
	111	101
Mean bit value	7/15	5/15

Mean bit

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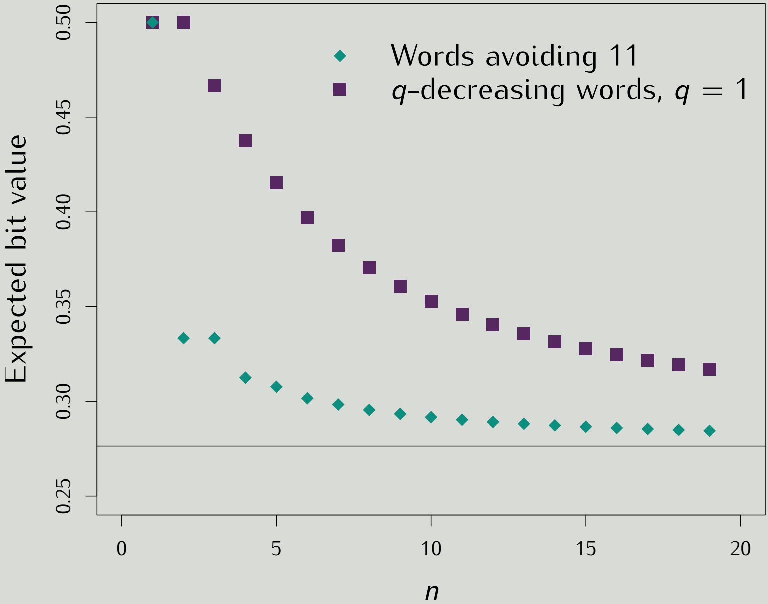
	1-decreasing	avoiding 11
	000	000
	001	001
	100	010
	110	100
	111	101
Mean bit value	7/15	5/15

- However, the mean bit values have common limit !

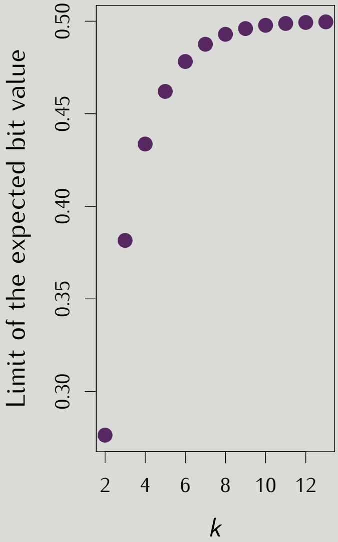
$$\left.\frac{kx^k - kx^{k-1} - x^k + 1}{kx^k - kx^{k-1} + x^{2k} - 3x^k + 2}\right|_{x=1/\varphi_k} \quad \text{when } n \to \infty,$$

where φ_k is the generalized golden ratio, φ_2 is the golden ratio, and $k = q + 1$.

Compare mean bit values in Fibonacci and q-decreasing words



Numerical limit of the mean bit value



k	Lim. mean bit val.
2	0.2763
3	0.3815
4	0.4336
5	0.4620
6	0.4782
7	0.4875
8	0.4929
9	0.4960
10	0.4977
11	0.4987
12	0.4993
13	0.4996

More about q-decreasing words

- Bijection between q-decreasing and Fibonacci words.
- Efficient generation and Gray codes
- Solved Eğecioğlu-Iršič conjecture
  (Hamiltonian path always exists in Fibonacci-run graphs).
- One proposed conjecture :)

Gray codes for Fibonacci q-decreasing words
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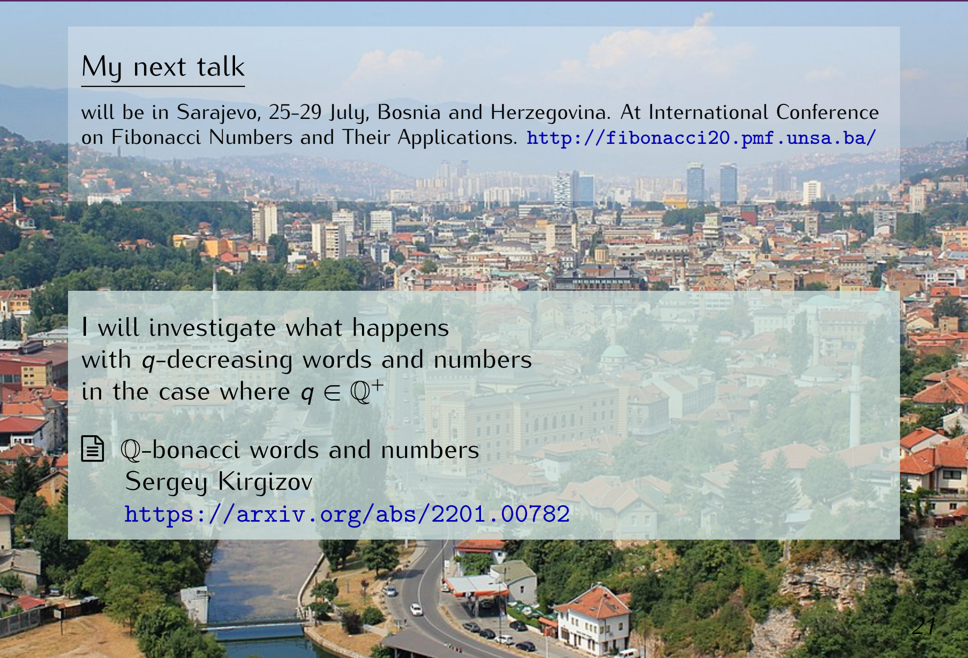
Grazie!



My next talk

will be in Sarajevo, 25-29 July, Bosnia and Herzegovina. At International Conference on Fibonacci Numbers and Their Applications. http://fibonacci20.pmf.unsa.ba/

I will investigate what happens
with q-decreasing words and numbers
in the case where $q \in \mathbb{Q}^+$

$\mathbb{Q}$-bonacci words and numbers
Sergey Kirgizov
https://arxiv.org/abs/2201.00782